Bayes Nets Representation: joint distribution and conditional independence

Yi Zhang
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Parts of the slides are from previous 10-701 lectures

Outline

- Conditional independence (C. I.)
- Bayes nets: overview
- Local Markov assumption of BNs
- Factored joint distribution of BNs
- Infer C. I. from factored joint distributions
- D-separation (motivation)

Conditional independence

- X is conditionally independent of Y given Z

$$(\forall x, y, z) P(X = x | Y = y, Z = z) = P(X = x | Z = z)$$

  - In short:

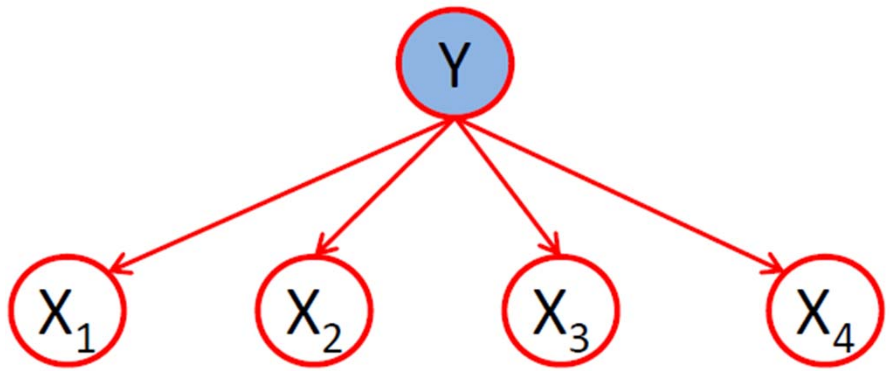
$$P(X \mid Y, Z) = P(X \mid Z)$$

  - Equivalent to:

$$P(X, Y \mid Z) = P(X \mid Z) P(Y \mid Z)$$

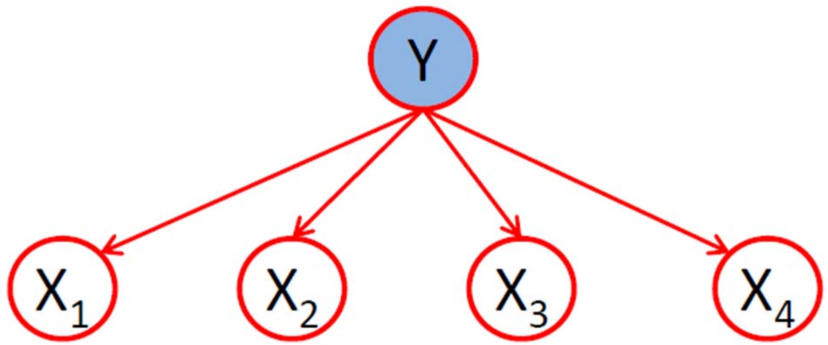
Bayes nets

- Bayes nets: directed acyclic graphs express sets of conditional independence via graph structure
  - All about the joint distribution of variables !
  - Conditional independence assumptions are useful
  - Naïve Bayes model is an extreme example



Three key questions for BNs

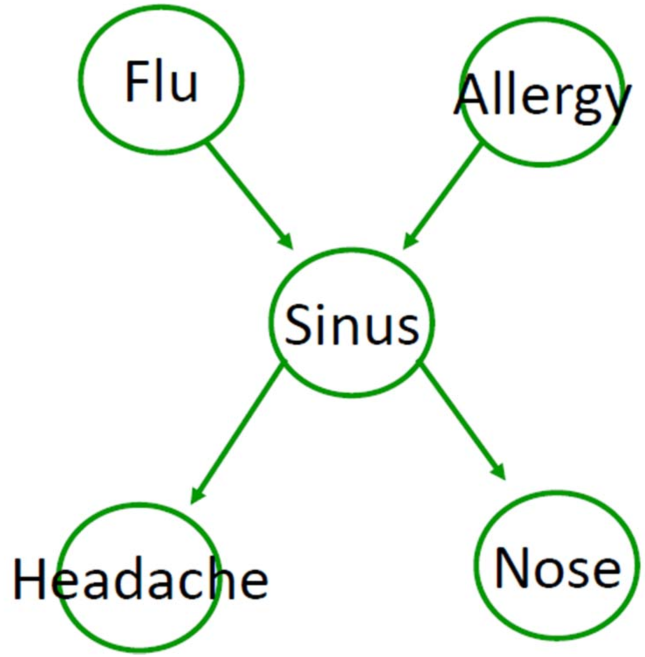
- ***Representation***:
  - ***What joint distribution does a BN represent?***
- Inference
  - How to answer questions about the joint distribution?
    - Conditional independence
    - Marginal distribution
    - Most likely assignment



- Learning
  - How to learn the graph structure and parameters of a BN from data?

Local Markov assumptions of BNs

- A variable X is independent of its non-descendants given (only) its parents
  - Intuition: "flu" and "allergy" causes "headache" only through "sinus"

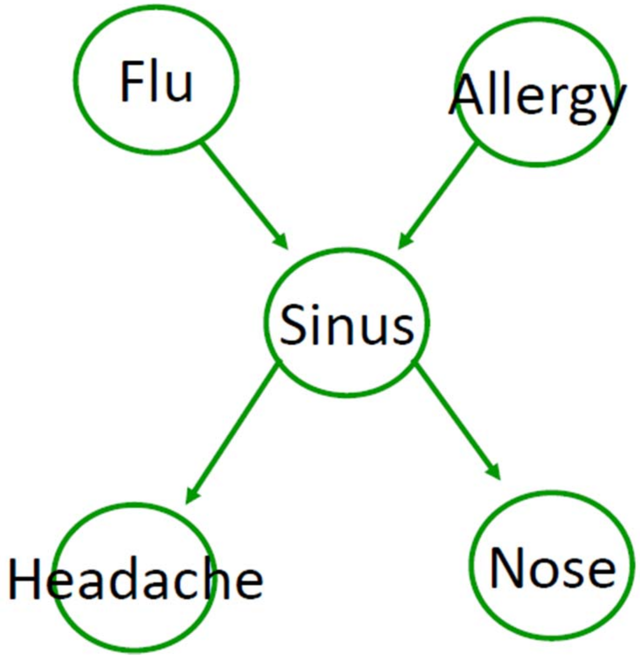


Local Markov assumptions of BNs

- A variable X is independent of its non-descendants given (only) its parents

	parents	non-desc	assumption
S	F,A	-	-
H	S	F,A,N	$H \perp \{F,A,N\} \mid S$
N	S	F,A,H	$N \perp \{F,A,H\} \mid S$
F	-	A	$F \perp A$
A	-	F	$A \perp F$

$F \perp A, \quad H \perp \{F,A\} \mid S, \quad N \perp \{F,A,H\} \mid S$



Local Markov assumptions of BNs

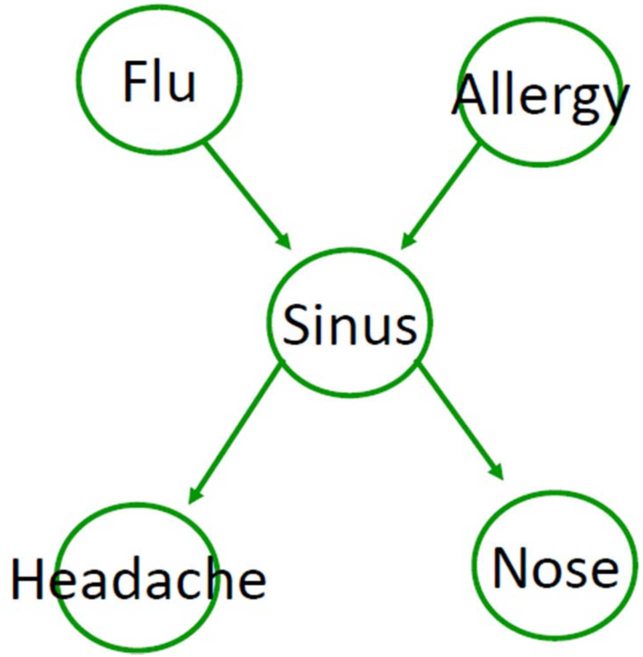
- Local Markov assumptions only express a ***subset*** of C.I.s on a BN
  - Is X_M conditionally independent of X_1 given X_2?

- But they are sufficient to infer all others

Factored joint distribution of a BN

- A BN can represent the joint distributions of the following form:

$$p(\mathbf{x}) = \prod_{k=1}^{K} p(x_k|\mathrm{pa}_k)$$



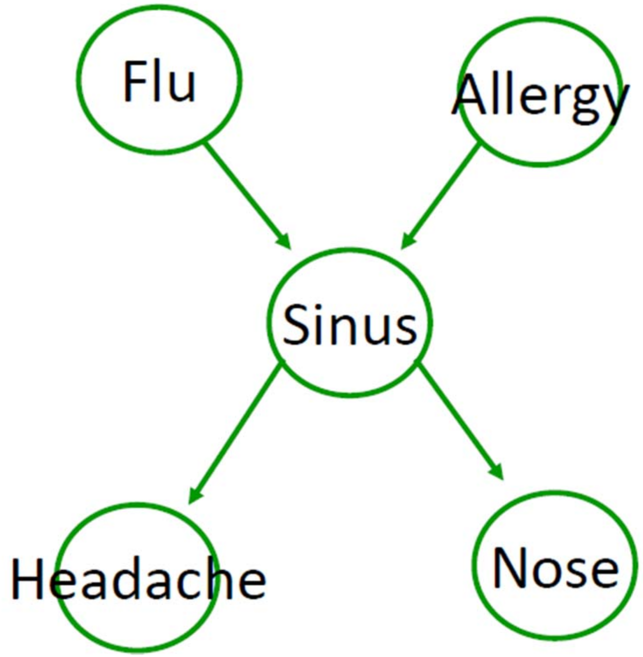
Factored joint distribution of a BN

- A BN can represent the joint distributions of the following form:

$$p(\mathbf{x}) = \prod_{k=1}^{K} p(x_k|\mathrm{pa}_k)$$

P(F, A, S, H, N)

= P(F) P(A) P(S|F,A) P(H|S) P(N|S)



Factored joint distribution of a BN

- Local Markov assumptions imply the factored joint distribution

$P(F, A, S, H, N)$

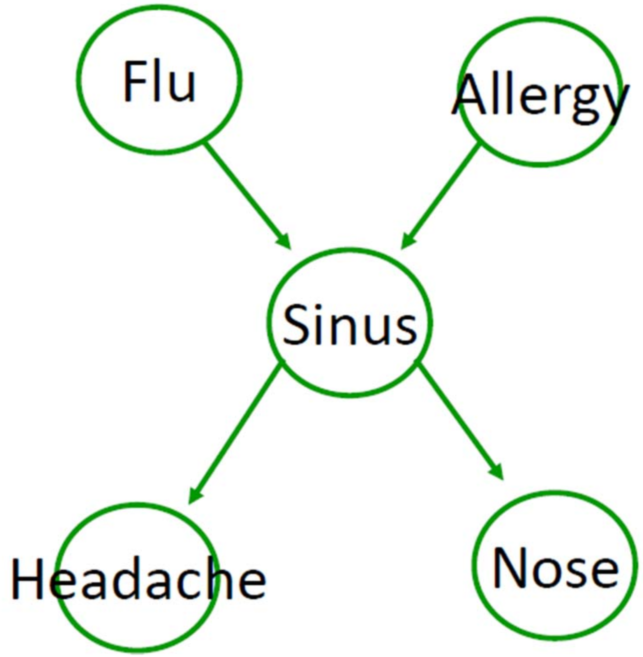
$= P(F)\, P(F|A)\, P(S|F,A)\, P(H|S,F,A)\, P(N|S,F,A,H)$

Chain rule

$= P(F)\, P(A)\, P(S|F,A)\, P(H|S)\, P(N|S)$

Markov Assumption

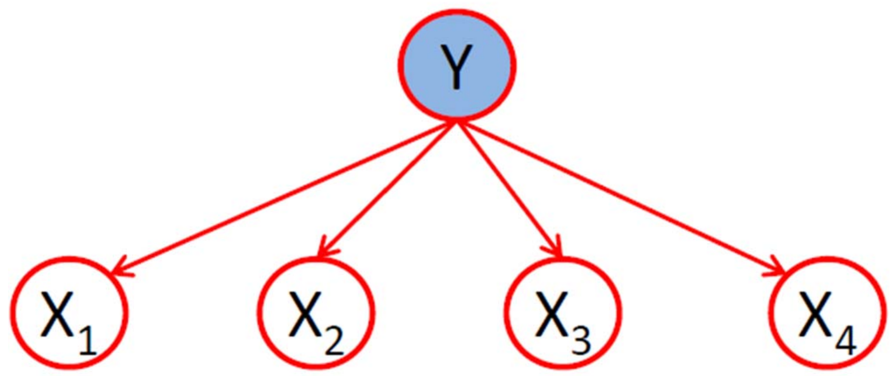
$F \perp A, \quad H \perp \{F,A\}|S, \quad N \perp \{F,A,H\}|S$



Factored joint distribution of a BN

- Naïve Bayes
  - Local Markov assumptions: $X_i \perp X_1,\ldots,X_{i-1},X_{i+1},\ldots, X_n | Y$
  - Factored joint distribution:

$$P(X_1,\ldots,X_n,Y) =$$
$$P(Y)P(X_1|Y)\ldots P(X_1|Y)$$



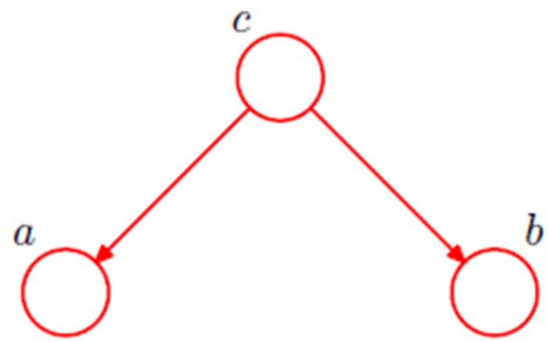
Infer C.I. from the factored joint distribution

- We already see: local Markov assumptions → factored joint distribution
- Also, factored joint distribution → all C.I. in the BN

Infer C.I. from the factored joint distribution

- Factored Joint distribution

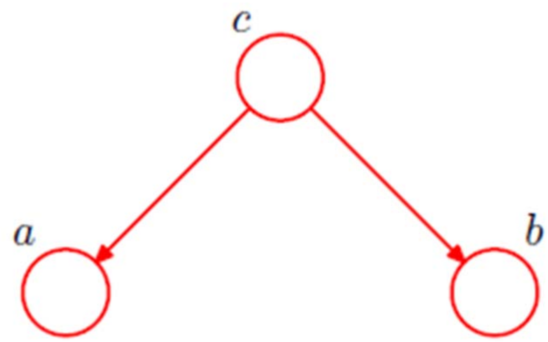
$$p(a, b, c) = p(a|c)p(b|c)p(c)$$



- Show that $a \perp\!\!\!\perp b \mid c$

$$\begin{aligned} p(a, b|c) &= \frac{p(a, b, c)}{p(c)} \\ &= \frac{p(a|c)p(b|c)p(c)}{p(c)} \\ &= p(a|c)p(b|c) \end{aligned}$$

Infer C.I. from the factored joint distribution



- Factored Joint distribution

$$p(a,b,c) = p(a|c)p(b|c)p(c)$$

- Do we have $a \perp\!\!\!\perp b$? In general, no.

$$p(a,b) = \sum_c p(a,b,c)$$
$$= \sum_c p(a|c)p(b|c)p(c)$$

  - Cannot be written into two separate terms of *a* and *b*

D-separation: motivation

- Is X_M conditionally independent of X_1 given X_2?

  - Intuitively yes: X_1 affects X_M only through X_2 .
  - Method I: using factored joint distribution to derive

$$p(x_1, x_M | x_2) = \frac{p(x_1, x_2, x_M)}{p(x_2)}$$

$$= \frac{\sum_{x_3, x_4, \ldots, x_{M-1}} p(x_1, x_2, \ldots, x_M)}{\sum_{x_1, x_3, x_4, \ldots, x_{M-1}, x_M} p(x_1, x_2, \ldots, x_M)}$$

  - Method II: D-separation ☺ --- not today