Machine Learning 10-601

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Today:

- Graphical models
- Bayes Nets:
  - Representing distributions
  - Conditional independencies
  - Simple inference
  - Simple learning

Readings:

Required:
- Bishop chapter 8, through 8.2

Graphical Models

- Key Idea:
  - Conditional independence assumptions useful
  - but Naïve Bayes is extreme!
  - Graphical models express sets of conditional independence assumptions via graph structure
  - Graph structure plus associated parameters define *joint probability distribution over set of variables*
- Two types of graphical models:
  - Directed graphs (aka Bayesian Networks) ← today
  - Undirected graphs (aka Markov Random Fields)

Graphical Models – Why Care?

- Among most important ML developments of the decade
- Graphical models allow combining:
  - Prior knowledge in form of dependencies/independencies
  - Prior knowledge in form of priors over parameters
  - Observed training data
- Principled and ~general methods for
  - Probabilistic inference
  - Learning
- Useful in practice
  - Diagnosis, help systems, text analysis, time series models, ...

Conditional Independence

Definition: X is conditionally independent of Y given Z, if the probability distribution governing X is independent of the value of Y, given the value of Z

$$(\forall i, j, k) P(X = x_i | Y = y_j, Z = z_k) = P(X = x_i | Z = z_k)$$

Which we often write $P(X|Y, Z) = P(X|Z)$

E.g., $P(Thunder|Rain, Lightning) = P(Thunder|Lightning)$

Marginal Independence

Definition: X is marginally independent of Y if

$$(\forall i, j) P(X = x_i, Y = y_j) = P(X = x_i) P(Y = y_j)$$

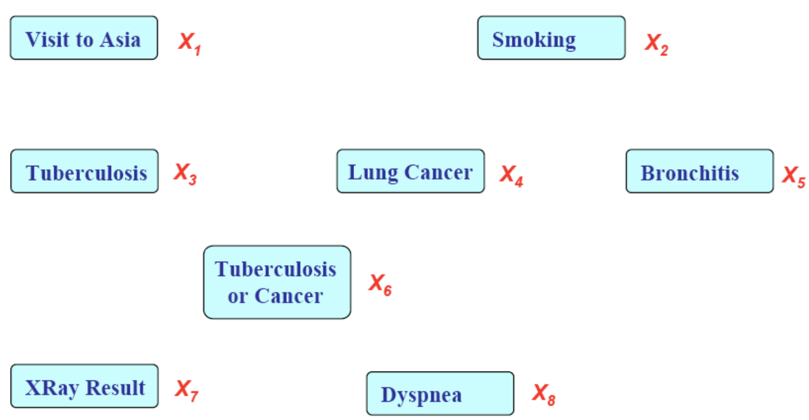
Equivalently, if

$$(\forall i, j) P(X = x_i | Y = y_j) = P(X = x_i)$$

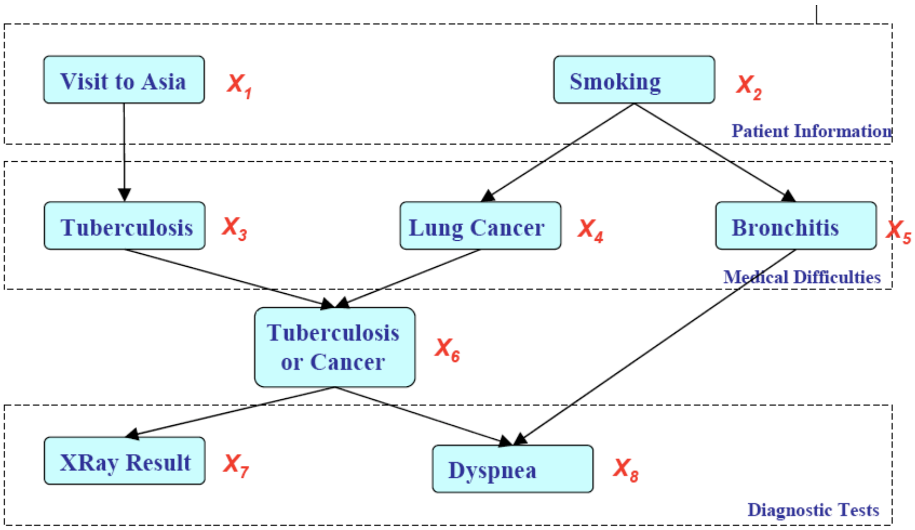
Equivalently, if

$$(\forall i, j) P(Y = y_i | X = x_j) = P(Y = y_i)$$

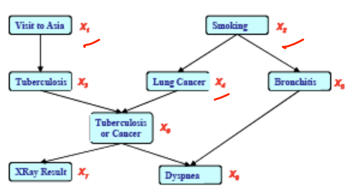
Represent Joint Probability Distribution over Variables



Describe network of dependencies



Bayes Nets define Joint Probability Distribution in terms of this graph, plus parameters

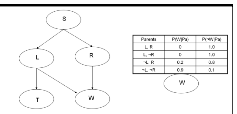


$$P(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)$$
$$= P(X_1)\, P(X_2)\, P(X_3 | X_1)\, P(X_4 | X_2)\, P(X_5 | X_2)\, P(X_6 | X_3, X_4)\, P(X_7 | X_6)\, P(X_8 | X_5, X_6)$$

Benefits of Bayes Nets:

- Represent the full joint distribution in fewer parameters, using prior knowledge about dependencies
- Algorithms for inference and learning

Bayesian Networks Definition



A Bayes network represents the joint probability distribution over a collection of random variables

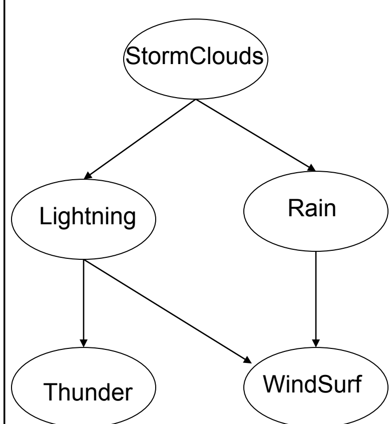
A Bayes network is a directed acyclic graph and a set of conditional probability distributions (CPD's)

- Each node denotes a random variable
- Edges denote dependencies
- For each node X_i its CPD defines $P(X_i \mid Pa(X_i))$
- The joint distribution over all variables is defined to be

$$P(X_1 \ldots X_n) = \prod_i P(X_i | Pa(X_i))$$

Pa(X) = immediate parents of X in the graph

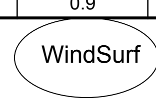
Bayesian Network



Nodes = random variables

A conditional probability distribution (CPD) is associated with each node N, defining P(N | Parents(N))

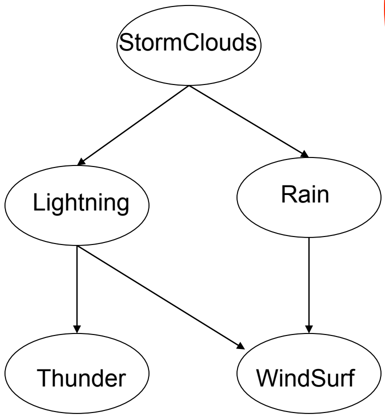
Parents	P(W|Pa)	P(¬W|Pa)
L, R	0	1.0
L, ¬R	0	1.0
¬L, R	0.2	0.8
¬L, ¬R	0.9	0.1



The joint distribution over all variables:

$$P(X_1 \ldots X_n) = \prod_i P(X_i | Pa(X_i))$$

Bayesian Network



What can we say about conditional independencies in a Bayes Net?

One thing is this:

Each node is conditionally independent of its non-descendents, given only its immediate parents.

Parents	P(W|Pa)	P(¬W|Pa)
L, R	0	1.0
L, ¬R	0	1.0
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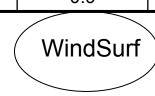


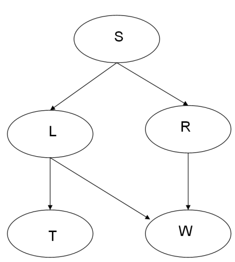
Some helpful terminology

Parents = Pa(X) = immediate parents

Antecedents = parents, parents of parents, ...

Children = immediate children

Descendents = children, children of children, ...



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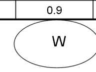
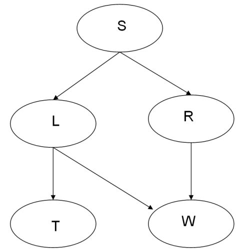


Bayesian Networks

- CPD for each node X_i describes $P(X_i \mid Pa(X_i))$



Parents	P(W|Pa)	P(¬W|Pa)
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W

$P(A,B) = P(A)\,P(B|A)$

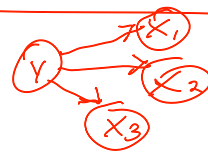
Chain rule of probability says that in general:

$$P(S,L,R,T,W) = P(S)P(L|S)P(R|S,L)P(T|S,L,R)P(W|S,L,R,T)$$

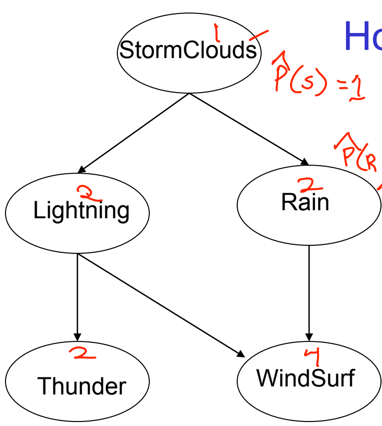
But in a Bayes net: $P(X_1 \ldots X_n) = \prod_i P(X_i | Pa(X_i))$

$P(S,L,R,T,W) = P(S)\,P(L|S)\,P(R|S)\,P(T|L)\,P(W|L,R)$

$X_1 \; X_2 \; X_3 \; Y$



How Many Parameters?



$P(S) = 1$

$P(R=1|S=1)$

$P(R=1|S=0)$

Parents	P(W|Pa)	P(¬W|Pa)
L, R	0	1.0
L, ¬R	0	1.0
¬L, R	0.2	0.8
¬L, ¬R	0.9	0.1

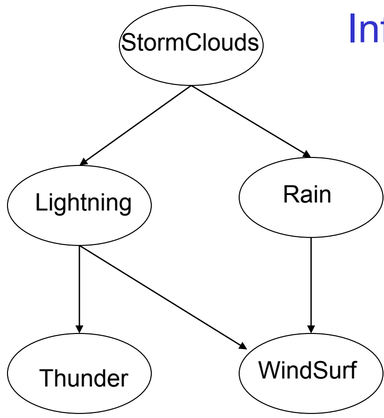
WindSurf

To define joint distribution in general? $2^5 - 1 = 31$

To define joint distribution for this Bayes Net? $= 11$

N Bayes (S → L, R, W, T) = 9 param

Inference in Bayes Nets



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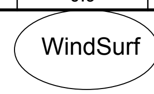
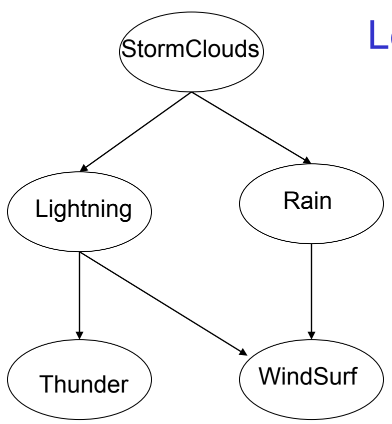


P(S=1, L=0, R=1, T=0, W=1) = P(S=1) P(L=0|S=1) P(R=1|S=1) P(T=0|L=0) P(W=1|L=0,R=1)

Learning a Bayes Net



P(W=1|Pa)

Parents	P(W|Pa)	P(¬W|Pa)
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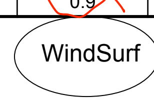


Consider learning when graph structure is given, and data = { <s,l,r,t,w> }

What is the MLE solution? MAP?

Algorithm for Constructing Bayes Network

- Choose an ordering over variables, e.g., $X_1, X_2, \ldots X_n$
- For i=1 to n
  - Add X_i to the network
  - Select parents $Pa(X_i)$ as minimal subset of $X_1 \ldots X_{i-1}$ such that

$$P(X_i|Pa(X_i)) = P(X_i|X_1, \ldots, X_{i-1})$$

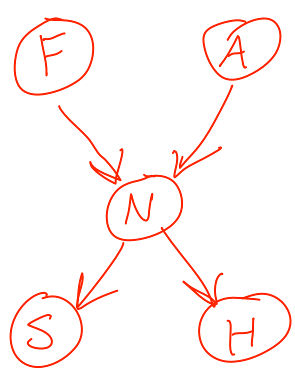
Notice this choice of parents assures

$$P(X_1 \ldots X_n) = \prod_i P(X_i|X_1 \ldots X_{i-1}) \quad \text{(by chain rule)}$$

$$= \prod_i P(X_i|Pa(X_i)) \quad \text{(by construction)}$$

Example

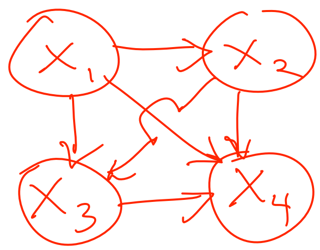
- Bird flu and Allegies both cause Nasal problems
- Nasal problems cause Sneezes and Headaches



$S \perp F \mid N$

$A \perp F$

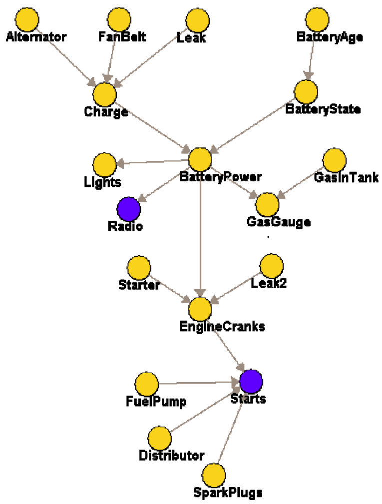
What is the Bayes Network for X1,…X4 with NO assumed conditional independencies?

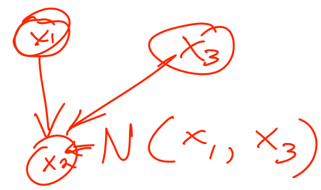


$$P(x_1\, x_2\, x_3\, x_4) = P(x_1)\, P(x_2|x_1)\, P(x_3|x_1, x_2)\, P(x_4|x_1 x_2 x_3)$$

What is the Bayes Network for Naïve Bayes?

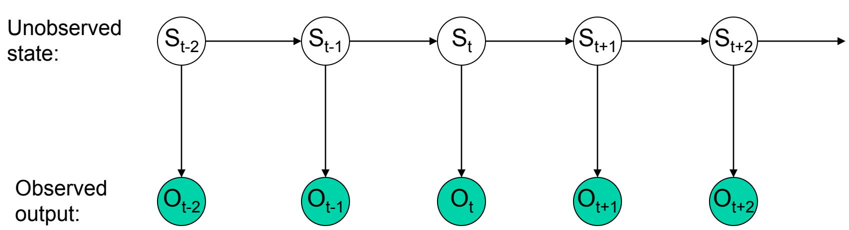
What do we do if variables are mix of discrete and real valued?





Bayes Network for a Hidden Markov Model

Implies the future is conditionally independent of the past, given the present



$$P(S_{t-2}, O_{t-2}, S_{t-1}, \ldots, O_{t+2}) =$$

What You Should Know

- Bayes nets are convenient representation for encoding dependencies / conditional independence
- BN = Graph plus parameters of CPD's
  - Defines joint distribution over variables
  - Can calculate everything else from that
  - Though inference may be intractable
- Reading conditional independence relations from the graph
  - Each node is cond indep of non-descendents, given only its parents
  - 'Explaining away'

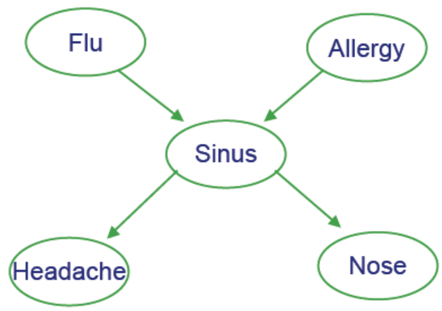
See Bayes Net applet: http://www.cs.cmu.edu/~javabayes/Home/applet.html

Inference in Bayes Nets

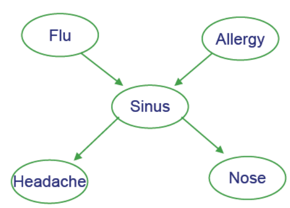
- In general, intractable (NP-complete)
- For certain cases, tractable
  - Assigning probability to fully observed set of variables
  - Or if just one variable unobserved
  - Or for singly connected graphs (ie., no undirected loops)
    - Belief propagation
- For multiply connected graphs
    - Junction tree
- Sometimes use Monte Carlo methods
  - Generate many samples according to the Bayes Net distribution, then count up the results
- Variational methods for tractable approximate solutions

Example

- Bird flu and Allegies both cause Sinus problems
- Sinus problems cause Headaches and runny Nose



Prob. of joint assignment: easy

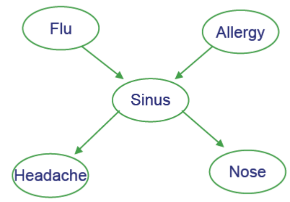


- Suppose we are interested in joint assignment <F=f,A=a,S=s,H=h,N=n>

What is P(f,a,s,h,n)?

let's use p(a,b) as shorthand for p(A=a, B=b)

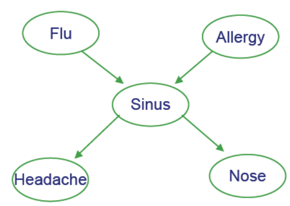
Prob. of marginals: not so easy



- How do we calculate P(N=n) ?

let's use p(a,b) as shorthand for p(A=a, B=b)

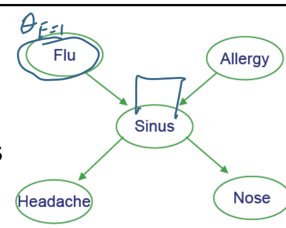
Generating a sample from joint distribution: easy



How can we generate random samples drawn according to P(F,A,S,H,N)?

let's use p(a,b) as shorthand for p(A=a, B=b)

Generating a sample from joint distribution: easy



How can we generate random samples drawn according to P(F,A,S,H,N)?

randomly draw a value for $F=f$

draw $r \in [0,1]$ uniformly

if $r < \theta_{F=1}$ then output $f=1$

else $f=0$

draw f, a, $s|f,a$, $h|s$, $n|s$

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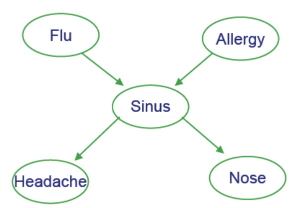
Generating a sample from joint distribution: easy



Note we can estimate marginals like P(N=n) by generating many samples from joint distribution, then count the fraction of samples for which N=n

Similarly, for anything else we care about

P(F=1|H=1, N=0)

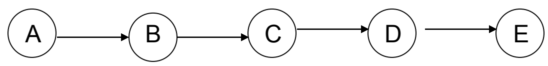
→ weak but general method for estimating <u>any</u> probability term…

let's use p(a,b) as shorthand for p(A=a, B=b)

Prob. of marginals: not so easy

But sometimes the structure of the network allows us to be clever → avoid exponential work

eg., chain



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