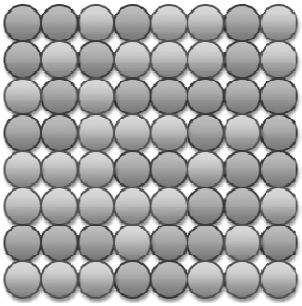


UNIT 14C

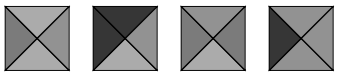
The Limits of Computing: Non-computable Functions

Problem Classifications

- Tractable Problems
  - Problems that have reasonable, polynomial-time solutions
- Intractable Problems
  - Problems that may have no reasonable, polynomial-time solutions
- Noncomputable Problems
  - Problems that have no algorithms at all to solve them

Non-computability

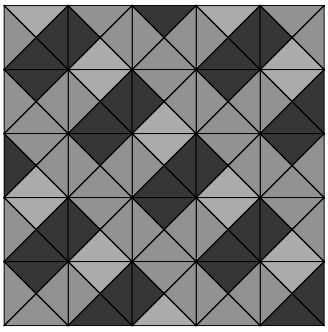
- A computational problem that has no general algorithm to solve any instance of the problem is called **noncomputable**.
  - If the noncomputable algorithm requires a yes/no answer, the problem is called **undecidable.**
- Example:



  - Given any set of any number of different tile designs (examples shown above), with an infinite number of each type of tile, can we tile any area with these tiles so that like colored edges touch?
  - This problem is undecidable!

Tiling

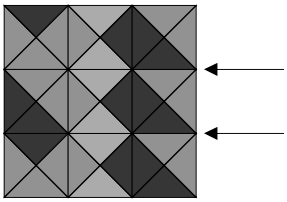
YES



Note the periodicity in the tiling.

Tiling

NO



For this 3 X 3 room, if we try all 3^9 tiling configurations, no tiling works.

Tiling

- Possible algorithm:
  - If we find a repeating pattern, report YES.
  - If we find a floor we cannot tile, report NO.
- BUT: there are some tilings which have no repeating pattern!

- The only way to know if this set of tiles can tile every finite-sized floor is to evaluate every possible floor.
- BUT: there are an infinite number of finite-sized floors!
  - So we could never answer YES in this case.

The Barber Paradox

Suppose there is a town with one male barber; and that every man in the town keeps himself clean-shaven: some shave themselves and some are shaved by the barber. Only the barber can shave another man. The barber shaves all and only those men who do not shave themselves.

Does the barber shave himself?

Program Termination

- Can we determine if a program will terminate given a valid input?
- Example:

```
def mystery1(x)
  while (x != 1) do
    x = x - 2
  end
end
```

  - Does this algorithm terminate when x = 15?
  - Does this algorithm terminate when x = 110?

Another Example

```
def mystery2(x)
  while (x != 1) do
    if x % 2 == 0 then
      x = x / 2
    else
      x = 3 * x + 1
  end
end
```

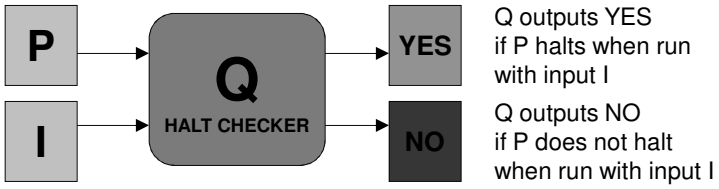
- Does this algorithm terminate when x = 15?
- Does this algorithm terminate when x = 110?
- Does this algorithm terminate for any positive x?

The Halting Problem

- Does a universal program Q exist that can take any program P and any input I for program P and determine if P terminates/halts when run with input I?
- Alan Turing showed that such a universal program Q cannot exist.
  - This is known as the Halting Problem.

Proof by Contradiction

- Assume a program Q exists that requires a program P and an input I.
  - Q determines if program P will halt when P is executed using input I.



  - We will show that Q cannot exist by showing that if it did exist we would get a logical contradiction.

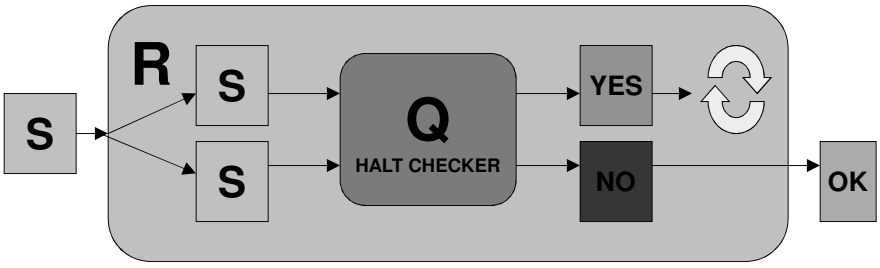
Compilers

- A compiler is a program that takes as its input a program that needs to be translated from a high-level language (e.g. Ruby) to a low-level language (e.g. machine language).
  - In general, a program can process any data, so it can have a program as its input to process.
- Can a compiler compile itself?

Proof (cont'd)

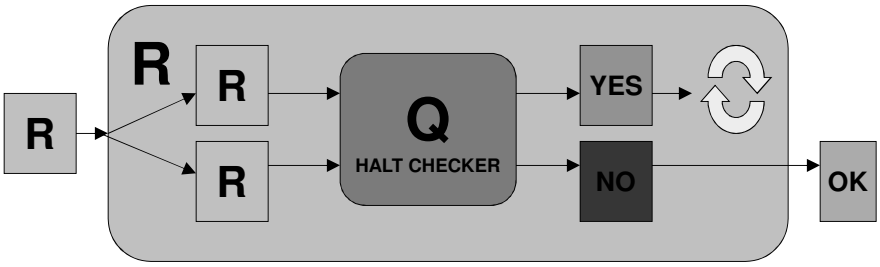
- Let R be a program that takes input S, where S is a program.
- R asks the halt checker Q what happens if S runs with itself as input?
- If Q answers that S will halt if it runs with itself as input, then R goes into an infinite loop (and does not halt).
- If Q answers that S will not halt if it runs with itself as input, then R halts.

How R Works



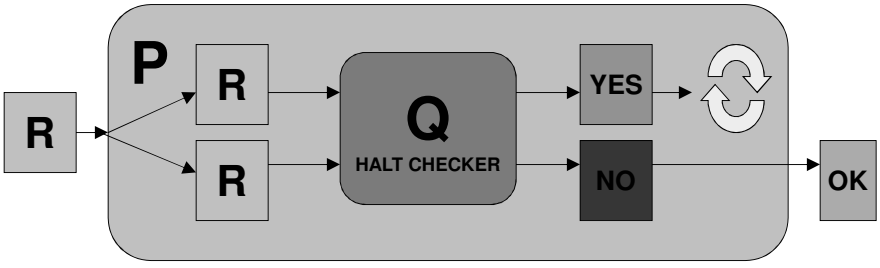
R gets evil

- What happens if R tests itself?
  - If Q answers yes (R halts), then R goes into an infinite loop and does not halt.



R gets evil

- What happens if R tests itself?
  - If Q answers no (R does not halt), then R halts.



Contradiction

- No matter what Q answers about R, R does the opposite, so Q can never answer the halting problem for the specific program R.
  - Therefore, a universal halting checker Q cannot exist.
- We can never write a computer program that determines if ANY program halts with ANY input.
  - It doesn't matter how powerful the computer is.
  - It doesn't matter how much time we devote to the computation.

Contradiction in Real Life