Compositional Verification of Cost and Behavior in Type Theory

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Motivation

Composition is the only game in town for building programs:

> let X : *interface* **be** *impl* **in** *client*

In lieu of verification X is held wholly abstract to permit change of implementation and help isolate errors.

Specification of cost and behavior is informal, if present at all, and not verified.

(Types do go a long way, though!)

Queues, Typically

A "typical" interface for queues in familiar settings:

interface PreQueue

X : **Type**

emp : X *(* no elements, O(1) *)*

enq : $\mathbb{N} \to X \to X$ *(* add to back, O(n) *)*

deq : $X \to \mathbb{N} \times X$ *(* take from front, O(1) *)*

Comments have no force ... and are often obsolete or omitted.

Bounds are worst-case? Stated as a function of what?

Compositional Verification?

For verification hiding information is a non-starter:

- Representation type cannot be hidden because ...
- ... specification of behavior requires it, and ...
- ... cost cannot be determined without behavior!

Lessons of history suggest, nothing short of a full implementation will suffice.

(cf experience with algebraic specifications)

Queues, Fully Specified

Eg, behavior of queues might be specified in terms of lists:

interface $\textsc{Queue}$

X : **Type** $= \textsc{List}\ \mathbb{N}$

empty : $X = []$

enqueue : $\mathbb{N} \to X \to X = \lambda n\ q \to q +\!\!+ [n]$

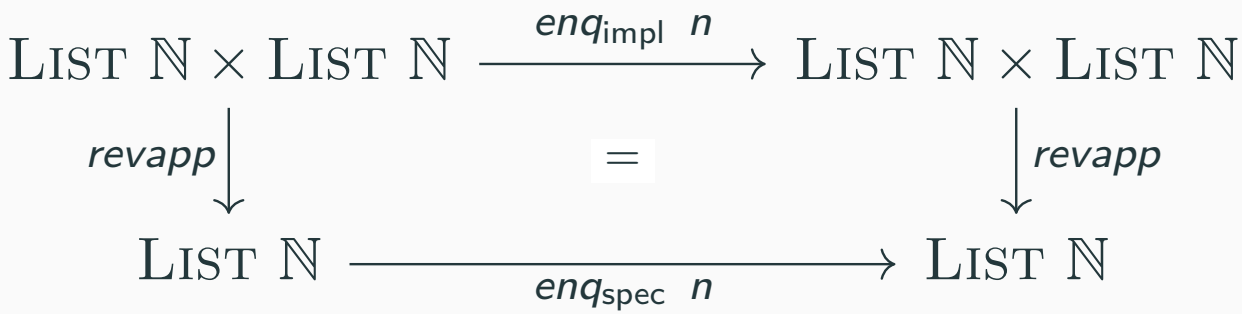
dequeue : $X \to \mathbb{N} \times X = \lambda q \to \ldots$

Client has a full understanding of the "abstraction" ...but it admits *only one* implementation!

And what about cost? Is linear time for enqueue now demanded?

Abstraction Functions

Hoare proposed abstraction functions to connect efficient to reference implementation



The abstraction function *revapp* sends (b, f) to $b \mathbin{@} rev(f)$.

Commutativity expresses correctness of the two-list implementation of enqueue relative to the specification.

$$revapp \circ (enq_{impl}\ n) = (enq_{spec}\ n) \circ revapp$$

(Each specification admits a cone of implementations over it.)

Abstraction Functions, with Cost

Hoare's method scales to account for cost, a la Calf and Decalf:

$$\begin{array}{ccc} \textsc{List}\ \mathbb{N} \times \textsc{List}\ \mathbb{N} & \xrightarrow{enq_{\text{impl}}\ n} & \mathbf{U}(\mathbf{F}(\textsc{List}\ \mathbb{N} \times \textsc{List}\ \mathbb{N})) \\ \Big\downarrow{\scriptstyle revapp} & \geq & \Big\downarrow{\scriptstyle \mathbf{U}(\mathbf{F}(revapp))} \\ \textsc{List}\ \mathbb{N} & \xrightarrow[enq_{\text{spec}}\ n]{} & \mathbf{U}(\mathbf{F}(\textsc{List}\ N)) \end{array}$$

Pre-order expresses upper bound on cost, but exact behavior.

$$\mathbf{U}(\mathbf{F}(revapp)) \circ (enq_{impl}\ n) \leq (enq_{spec}\ n) \circ revapp$$

Synthetic Formulation

Every type X is (tacitly) given by an abstraction function.

- No loss of generality: choose the identity map.
- Full range of dependent type theory constructs is available ...
- ... augmented by glueing for examples such as queues.

Synthetic formulation uses two modalities induced by a phase:

- Abstract, $\bigcirc X$: isolates specification.
- Concrete, $\bullet X$: isolates implementation.

(cf behavioral phase to suppress cost in Calf and Decalf)

Queues, Reference Interface w/Cost

Behavioral specification is augmented with (implausible!) costs:

interface $\textsc{Queue}$

$X : \mathbf{Type} \mathrel{\hat{=}} \textsc{List}\ \mathbb{N}$

$\mathsf{emp} : X \mathrel{\hat{=}} \mathbf{charge}\langle 1\rangle; []$

$\mathsf{enq} : \mathbb{N} \to X \to X \mathrel{\hat{=}} \lambda n\ q \to \mathbf{charge}\langle 1\rangle; q +\!\!+ [n]$

$\mathsf{deq} : X \to \mathbb{N} \times X \mathrel{\hat{=}} \lambda q \to \mathbf{charge}\langle |q|\rangle; \ldots$

(Here $M \mathrel{\hat{=}} N$ means $\bigcirc(M = N)$, valid only in abstract phase.)

Cost is not empirical, it is rather specified by the code!

Synthetic Formulation: Glueing

The value type of queues is given by glueing:

$$\mathsf{Glue}^{\mathcal{V}}(\bullet(\textsc{List}\ \mathbb{N} \times \textsc{List}\ \mathbb{N}), \bigcirc(\textsc{List}\ \mathbb{N}), \bullet(\eta_{\circ} \circ \mathit{revapp}))$$

Modalities isolate concrete and abstract aspects of types, even for primitive types.

Operations are implemented by a sealing effect a la modules:

$$\mathit{seal}(\mathit{enq}_{\mathit{impl}}, \mathit{enq}_{\mathit{spec}}, \mathit{prf})$$

where prf is evidence of the inequality to match interface bound.

Queue Implementation

Formulated in an ML-like notation:

$$
\begin{aligned}
&\mathit{BatchedQ} : \textsc{Queue} \\
&\mathit{BatchedQ}.X := \textsc{List}\ \mathbb{N} \times \textsc{List}\ \mathbb{N} \\
&\quad \textbf{with}\ \alpha\ (\mathit{back}, \mathit{front}) := \mathit{front} +\!\!+ \mathit{rev}(\mathit{back}) \\
&\mathit{BatchedQ}.\mathsf{emp} := ([], []) \\
&\quad \textbf{with}\ \_ : \alpha\ (\mathit{BatchedQ}.\mathsf{emp}) \mathrel{\mathring{=}} \textsc{Queue}.\mathsf{emp} \\
&\mathit{BatchedQ}.\mathsf{enq}\ n\ (\mathit{back}, \mathit{front}) := \textbf{charge}\langle 1\rangle; (n :: \mathit{back}, \mathit{front}) \\
&\quad \textbf{with}\ \_ : \alpha\ (\mathit{BatchedQ}.\mathsf{enq}\ n\ q) \mathrel{\mathring{=}} \textsc{Queue}.\mathsf{enq}\ n\ (\alpha\ q) \\
&\mathit{BatchedQ}.\mathsf{deq}\ (\mathit{back}, n :: \mathit{front}) := \\
&\quad \textbf{charge}\langle 1\rangle; (n, (\mathit{back}, \mathit{front})) \\
&\mathit{BatchedQ}.\mathsf{deq}\ (\mathit{back}, []) := \\
&\quad \textbf{charge}\langle |\mathit{back}|\rangle; \mathit{BatchedQ}.\mathsf{deq}\ ([], \mathit{rev}(\mathit{back})) \\
&\quad \textbf{with}\ \_ : (\mathbb{N} \times \alpha)\ (\mathit{BatchedQ}.\mathsf{deq}\ q) \mathrel{\mathring{=}} \textsc{Queue}.\mathsf{deq}\ (\alpha\ q)
\end{aligned}
$$

Amortization?

Weak bound for dequeue is pessimistic, but cannot be avoided.

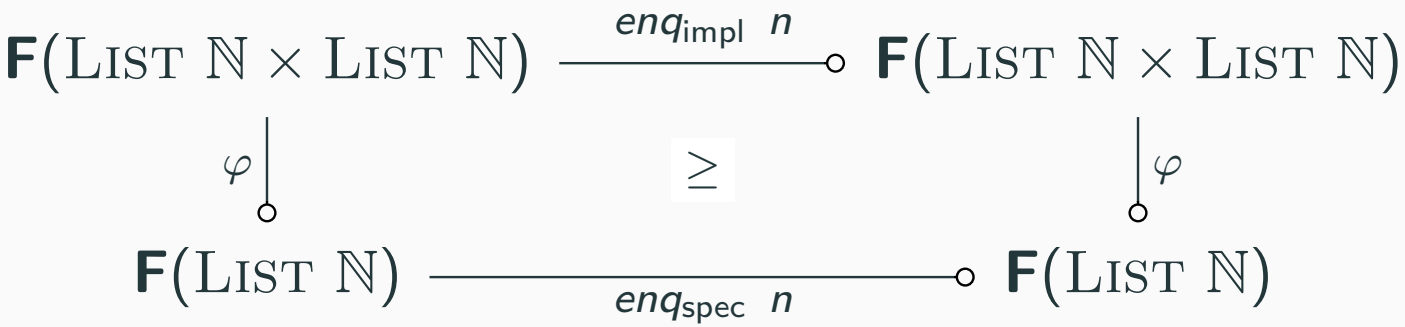
- In a structural setting queues may be used persistently, leading to replicated work.
- Rules out amortization, which relies on queues being ephemeral, allowing for eventual work.

To ensure ephemeral usage, extend Calf with a linear context at the computation level (a la EEC, LNL).

- Values remain persistent and otherwise unchanged.
- Computations may restrict usage via linear entailment.

Potential Functions as Types?

Potential functions enrich abstraction functions with cost:



where φ is the linear function

$$\lambda(\mathbf{ret}(b, f)) \multimap \mathbf{charge}\langle|b|\rangle; \mathbf{ret}(\mathit{revapp}(b, f)).$$

Linearity is used in both dimensions:

- Vertically, cost is stored potential.
- Horizontally, cost is incurred charge.

Lax commutativity expresses amortization equations relating cost and potential.

Amortized Specification Interface

Interface for ephemeral queues:

interface $\textsc{EQueue}$

$A : \mathbf{CType} \doteq \mathbf{F}(\textsc{List}\ \mathbb{N})$

$\mathsf{emp} : \mathbf{U}(A) \doteq \mathbf{charge}\langle 1\rangle; \mathbf{ret}([])$

$\mathsf{enq} : \mathbb{N} \times A \multimap A \doteq \lambda(n, \mathbf{ret}(q)) \multimap \mathbf{charge}\langle 1\rangle; q +\!\!+ [n]$

$\mathsf{deq} : A \multimap \mathbb{N} \times A \doteq \lambda q \multimap \mathbf{charge}\langle 1\rangle; \ldots$

Unit cost for dequeue reflects use of linearity to express amortized bounds.

Potential Functions, Synthetically

Extend computation types with $\bigcirc A$, $\bullet A$, and glueing to specify amortized interfaces.

$$\mathsf{Glue}^{\mathcal{C}}(\bullet \mathbf{F}(\textsc{List}\ \mathbb{N} \times \textsc{List}\ \mathbb{N}), \bigcirc \mathbf{F}(\textsc{List}\ \mathbb{N}), \bullet(\eta_{\circ} \circ \varphi))$$

The operations themselves remain (largely) unchanged, only the analysis differs!

Amortized Queues

Implementation of amortized queues:

$$
\begin{aligned}
&\mathit{BatchedQ} : \textsc{EQueue} \\
&\mathit{BatchedQ}.X := \mathbf{F}(\textsc{List}\ \mathbb{N} \times \textsc{List}\ \mathbb{N}) \\
&\quad \textbf{with}\ \varphi(\textbf{ret}(b, f)) \ := \textbf{charge}\langle |b| \rangle; \textbf{ret}(\mathit{revapp}(b, f)) \\
&\mathit{BatchedQ}.\mathsf{emp} := \textbf{ret}([], []) \\
&\quad \textbf{with}\ \_ : \varphi\ (\mathit{BatchedQ}.\mathsf{emp}) \mathrel{\dot{=}} \textsc{EQueue}.\mathsf{emp} \\
&\mathit{BatchedQ}.\mathsf{enq}\ n\ (\mathit{back}, \mathit{front}) := \textbf{charge}\langle 1 \rangle; (n :: \mathit{back}, \mathit{front}) \\
&\quad \textbf{with}\ \_ : \varphi\ (\mathit{BatchedQ}.\mathsf{enq}\ n\ q) \mathrel{\dot{=}} \textsc{EQueue}.\mathsf{enq}\ n\ (\varphi\ q) \\
&\vdots
\end{aligned}
$$

Ongoing Work

Variant of "big O" notation for $f, g : X \to \mathbb{N}$:

- Define $f \preceq g$ iff $f \leq c + k\,g$ for some $c \geq 0$ and $k > 0$.
- Smoothly accounts for multi-variate functions, other domains.

Define a modality, $\mathcal{O}(A)$, on computation types.

- Add **scale**$\langle k \rangle; e$ to **charge**$\langle c \rangle; e$ to express inequation.
- Rescaling is permitted (and sensible) only in abstract phase!

With this modality the informal interface becomes formal!

Ongoing Work

Interpret Hoffmann's AARA in Calf w/amortization.

- Use computation glueing to add potential to incurred costs.
- RAML as a tactic for automatic cost inference in Calf.

Verification of meta-theory.

- Calf itself: CPP 26.

Library of case studies:

- RBT's, Parallel Scan, Splay Tree Merge (PoPL 26).

Probabilistic bounds?

Summary

Compositional verification of cost and behavior!

But why only now?

- Univalence equates equivalent representations.
- Phases distinguish abstraction from implementation (and behavior from cost).
- Inequality of computations to express lax bounds.
- Linearity provides for amortization.

Thanks!

Calf web page for links to papers, case studies, mechanized meta-theory:

`https://calfproject.github.io/calf/`

(See the papers linked there for citations and comparisons to related work.)