Higher order focusing for ordered logic
Request For Logic (RFL) #7
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In the previous note, we talked about coverage checking and case analysis with linear functions representing one-hole contexts. In this note, we push this further to do a version of higher-order focusing for ordered logic using the same notion of linear functions for one-hole contexts that reach into structures. Ordered logic occupies a unique position in this respect: because any one-hole context ƛx.Γ'(x) into an ordered structure is equivalent to ƛx.Γ,x,Δ, we can get away with explicitly representing the left and right hand sides. Experience seems to show that this is often quite a bit less satisfying than in linear logic where a one-hole context λx.Γ'(x) is equivalent to ƛx.Δ,x - the formulation we use here seems, in any case, no worse than the explicit left-hand-side-right-hand-side formulation..

We only consider the Lambek calculus portion of ordered logic: that is, we don't consider the validity or mobility modalities. Our proof appears to have the property that if we changed the properties of the context formation operator "," we would still have a valid proof - in other words, this can be seen as a proof of rigid logic or a redundant proof of linear logic just by manipulating the algebraic properties of the comma.

Higher order focusing for ordered logic

Contexts are written as Γ or Δ. The context formation operator "," is associative and commutative with unit "·".

$$\Gamma,\Delta ::= \mathrm{hyp}\ A^- \mid \mathrm{hyp}\ Q^+ \mid \Gamma,\Delta \mid \cdot$$

We adopt the convention that a context with one hole is written as Γ' or Δ', a context with two holes is written as Γ'' or Δ'', and so on. However, these should really be understood as eta-contracted versions of the representational linear functions (ƛx.Γ'(x)), (ƛx.ƛy.Γ''(x)(y)), and so on. The hole [] is similarly shorthand for (ƛx.x).

The basic idea is that while we usually write the cut principle for ordered logic like this:

If $\Omega \vdash A$ and $\Omega_1, \mathrm{hyp}\ A, \Omega_2 \vdash C$, then $\Omega_1,\Omega,\Omega_2 \vdash C$

However, we can talk about this cut principle more generally with one-hole contexts Ω', where we write the cut principle like this:

If $\Omega \vdash A$ and $\Omega'(\mathrm{hyp}\ A) \vdash C$, then $\Omega'(\Omega) \vdash C$

Patterns

In higher-order focusing, propositions are handled by patterns, which are defined independently of the rules of the logic. Positive propositions are defined by constructor patterns, and negative propositions are defined by destructor patterns.

$$A^+ ::= Q^+ \mid {\downarrow}A^- \mid A^+ \bullet B^+ \mid 1 \mid A^+ \oplus B^+ \mid 0$$
$$A^- ::= Q^- \mid {\uparrow}A^+ \mid A^+ \twoheadrightarrow B^- \mid A^+ \rightarrowtail B^- \mid A^- \,\&\, B^- \mid \top$$
$$\gamma ::= Q^- \mid A^+$$

Constructor patterns

```
---------------
hyp Q⁺ ⊩ Q⁺

---------------
hyp A⁻ ⊩ ↓A⁻

Δ₁ ⊩ A₁
Δ₂ ⊩ A₂
-----------------
Δ₁,Δ₂ ⊩ A₁⁺ • A₂⁺


---------------
· ⊩ 1
Δ ⊩ A
---------------
Δ ⊩ A ⊕ B

Δ ⊩ B
---------------
Δ ⊩ A ⊕ B
```

Destructor patterns

Destructor patterns are interesting because their type is $((\text{ctx} \multimap \text{ctx}) \rightarrow \text{prop}^- \rightarrow \text{gamma} \rightarrow \text{type})$ - in other words, one of the outputs is not a context but rather a linear function from contexts to contexts (a context with a hole in it). When we write $\Delta'(\Delta_1[])$ we "really mean" the linear function $\check{\lambda}\Delta_2.\Delta'(\Delta_1\Delta_2)$.

```
Δ₁ ⊩ A₁⁺
Δ₂' ⊩ A₂⁻ > γ
--------------------------
Δ₂'(Δ₁,[]) ⊩ A₁⁺ ↣ A₂⁻ > γ

Δ₁ ⊩ A₁⁺
Δ₂' ⊩ A₂⁻ > γ
--------------------------
Δ₂'([],Δ₁) ⊩ A₁⁺ ↠ A₂⁻ > γ

Δ' ⊩ A > γ
--------------------------
Δ' ⊩ A & B > γ

Δ' ⊩ B > γ
--------------------------
Δ' ⊩ A & B > γ

--------------------------
[] ⊩ ↓A⁺ > A⁺

--------------------------
[] ⊩ Q⁻ > Q⁻
```

Properties of patterns

In order for cut elimination to terminate, we depend on the subformula property of patterns. The introduction of, for instance, recursive types where this does not hold may be reasonable programming languages, but can have nonterminating cut elimination procedures.

```
(S^+) If Δ ⊩ A^+ then size(Δ) < size(A^+)
(S^-) If Δ ⊩ A^- > γ, then size(Δ) < size(A^-) and size(γ) < size(A^-)
```

Rules of the focused logic

The only terribly curious part of the focused logic is our use of a representational linear function in the rule for having a one-hole context imply a one-hole context (the "holey alpha substitution" rule).

Left focus

```
Δ' ⊩ A^- > γ_o      ⇐ Figure out the pattern of A^-
Γ' ≡ Γo' ∘ Γi'      ⇐ Γ' is the composition of outside Γo' and inside Γi'
Γi' ⊢ Δ'            ⇐ Alpha substitution (= prove stuff, possibly after inverting)
Γo' ⊢ γ_o > γ       ⇐ Gamma substitution (= prove stuff, possibly after inverting)
--------------- Perform Left Focus
Γ' ⊢ [A^-] > γ

Γ ≡ Γ'(hyp A^-)
Γ' ⊢ [A^-] > γ
--------------- Enter Left Focus
Γ ⊢ γ
```

Right focus

```
Δ ⊩ A^+             ⇐ Figure out the pattern of A^+
Γ ⊢ Δ               ⇐ Alpha substitution (= prove stuff, possibly after inverting)
--------------- Perform Right Focus
Γ ⊢ [A^+]

Γ ⊢ [A^+]
--------------- Enter Right Focus
Γ ⊢ A^+
```

Inversion

```
Δ' ⊩ A^- > γ  ⇒  Δ'(Γ) ⊢ γ
-------------------------- Right inversion
Γ ⊢ A^-

Δ ⊩ A^+  ⇒  Γ'(Δ) ⊢ γ
-------------------------- Left inversion
Γ' ⊢ A^+ > γ

Γ' ≡ []
-------------------------- Atom
Γ' ⊢ Q^- > Q^-
```

Alpha substitution

```
Γ1 ⊢ Δ1
Γ2 ⊢ Δ2
--------------- Split
Γ1,Γ2 ⊢ Δ1,Δ2

--------------- Unit
· ⊢ ·

--------------- Atomic
hyp Q^+ ⊢ hyp Q^+

Γ ⊢ A^-
--------------- Invert and prove
Γ ⊢ hyp A^-
```

Holey alpha substitution

```
ΠΓ. ΠΔ. (Γ ⊢ Δ) -o Γ'(Γ) ⊢ Δ'(Δ)
---------------------------------- Holey
Γ' ⊢ Δ'
```

Identity

This should be true and straightforward, but we don't prove it here (Dan sketched it out on paper).

Cut principles

Cut is proved by a slew of mutually inductive statements, as usual. In all cases, the induction argument is either that the principal formula $A^+/A^-/\Delta/\Delta'$ gets smaller, or the principal formula stays the same and one of the input derivations gets smaller while the other stays the same.

(+)	If $\Gamma \vdash [A^+]$	and $\Gamma' \vdash A^+ > \gamma$	then $\Gamma'(\Gamma) \vdash \gamma$
(-)	If $\Gamma \vdash A^-$	and $\Gamma' \vdash [A^-] > \gamma$	then $\Gamma'(\Gamma) \vdash \gamma$
(R1)	If $\Gamma \vdash \Delta$	and $\Gamma'(\Delta) \vdash \gamma$	then $\Gamma'(\Gamma) \vdash \gamma$
(R2)	If $\Gamma_1' \vdash \Delta'$	and $\Gamma'(\Delta'(\Gamma)) \vdash \gamma$	then $\Gamma'(\Gamma_1'(\Gamma)) \vdash \gamma$
(R3)	If $\Gamma \vdash A^-$	and $\Gamma'(\text{hyp } A^-) \vdash \gamma$	then $\Gamma'(\Gamma) \vdash \gamma$
(R4)	If $\Gamma \vdash A^-$	and $\Gamma''(\text{hyp } A^-) \vdash [B^-] > \gamma$	then $\Gamma''(\Gamma) \vdash [B^-] > \gamma$
(R5)	If $\Gamma \vdash A^-$	and $\Gamma'(\text{hyp } A^-) \vdash [B^+]$	then $\Gamma'(\Gamma) \vdash [B^+]$
(R6)	If $\Gamma \vdash A^-$	and $\Gamma''(\text{hyp } A^-) \vdash \gamma_0 > \gamma$	then $\Gamma''(\Gamma) \vdash \gamma_0 > \gamma$
(R7)	If $\Gamma \vdash A^-$	and $\Gamma''(\text{hyp } A^-) \vdash \Psi'$	then $\Gamma''(\Gamma) \vdash \Psi'$
(R8)	If $\Gamma \vdash A^-$	and $\Gamma'(\text{hyp } A^-) \vdash \Psi$	then $\Gamma'(\Gamma) \vdash \Psi$
(R9)	If $\Gamma \vdash A^-$	and $\Gamma'(\text{hyp } A^-) \vdash B^-$	then $\Gamma'(\Gamma) \vdash B^-$
(L1)	If $\Gamma_1 \vdash \gamma_0$	and $\Gamma' \vdash \gamma_0 > \gamma$	then $\Gamma'(\Gamma_1) \vdash \gamma$
(L2)	If $\Gamma_1' \vdash [A^-] > \gamma_0$	and $\Gamma' \vdash \gamma_0 > \gamma$	then $\Gamma' \circ \Gamma_1' \vdash [A^-] > \gamma$
(L3)	If $\Gamma_1' \vdash \gamma_1 > \gamma_0$	and $\Gamma' \vdash \gamma_0 > \gamma$	then $\Gamma' \circ \Gamma_1' \vdash \gamma_1 > \gamma$

Proof of (+)

$$\frac{D_1 :: \Delta \Vdash A^+ \qquad D_2 :: \Gamma \vdash \Delta}{\Gamma \vdash [A^+]}\ \text{Perform Right Focus}$$

$$\frac{E :: \Delta \Vdash A^+ \;\Rightarrow\; \Gamma'(\Delta) \vdash \gamma}{\Gamma' \vdash A^+ > \gamma}\ \text{Left inversion}$$

By application of E and D_1, $E_1 :: \Gamma'(\Delta) \vdash \gamma$
By (S^+) on D_1 $\Delta < A^+$, so by (R1) on D_2 and E_1, $F_1 :: \Gamma'(\Gamma) \vdash \gamma$

Proof of (-)

$$\frac{D :: \Delta' \Vdash A^- > \gamma \;\Rightarrow\; \Delta'(\Gamma) \vdash \gamma}{\Gamma \vdash A^-}\ \text{Right inversion}$$

$$\frac{E_1 :: \Delta' \Vdash A^- > \gamma_0 \qquad E_2 :: \Gamma' \equiv \Gamma o' \circ \Gamma i' \qquad E_3 :: \Gamma i' \vdash \Delta' \qquad E_4 :: \Gamma o' \vdash \gamma_0 > \gamma}{\Gamma' \vdash [A^-] > \gamma}\ \text{Perform Left Focus}$$

By application of D and E_1, $D_1 :: \Delta'(\Gamma) \vdash \gamma_0$
Because $\Delta' < A^-$, by (R2) on E_3 and D_1, $F_1 :: \Gamma i'(\Gamma) \vdash \gamma_0$
Because $\gamma_0 < A^-$, by (L1) on F_1 and E_4, $F_2 :: \Gamma o'(\Gamma i'(\Gamma)) \vdash \gamma$
By equality E_2 using F_2, $F_3 :: \Gamma'(\Gamma) \vdash \gamma$

Proof of (R1)

Proof proceeds by case analysis on the first derivation.

Case 1:

$$\frac{D_1 :: \Gamma_1 \vdash \Delta_1 \qquad D_2 :: \Gamma_2 \vdash \Delta_2}{\Gamma_1,\Gamma_2 \vdash \Delta_1,\Delta_2}\ \text{Split}$$

$E :: \Gamma'(\Delta_1,\Delta_2) \vdash \gamma$

To show: $\Gamma'(\Gamma_1,\Gamma_2) \vdash \gamma$
Because $\Gamma'(\Delta_1,\Delta_2) = (\lambda x.\Gamma'(x,\Delta_2))(\Delta_1)$, by (R1) on D_1 and E we have
$F_1 :: \Gamma'(\Gamma_1,\Delta_2) \vdash \gamma$
Because $\Gamma'(\Gamma_1,\Delta_2) = (\lambda x.\Gamma'(\Gamma_1,x))(\Delta_2)$, by (R1) on D_2 and F_1 we have
$F_2 :: \Gamma'(\Gamma_1,\Gamma_2) \vdash \gamma$

Case 2:

$$\frac{}{\cdot \vdash \cdot}\ \text{Unit}$$

$E :: \Gamma'(\cdot) \vdash \gamma$

To show: $\Gamma'(\cdot) \vdash \gamma$. Immediate from E.

Case 3:

$$\frac{}{\mathrm{hyp}\ Q^{+} \vdash \mathrm{hyp}\ Q^{+}}\ \text{Atomic}$$

$E :: \Gamma'(\mathrm{hyp}\ Q^{+}) \vdash \gamma$

To show: $\Gamma'(\mathrm{hyp}\ Q^{+}) \vdash \gamma$. Immediate from E.

Case 4:

$$\frac{D :: \Gamma \vdash A^{-}}{\Gamma \vdash \mathrm{hyp}\ A^{-}}\ \text{Invert and prove}$$

$E :: \Gamma'(\mathrm{hyp}\ A^{-}) \vdash \gamma$

To show: $\Gamma'(\Gamma) \vdash \gamma$. By (R3) on D and E.

Proof of (R2)

The first proof rule is definitely Holey, so by inversion we have $D' :: \Pi\Gamma.\ \Pi\Delta.\ (\Gamma \vdash \Delta) \multimap (\Gamma'(\Gamma) \vdash \Delta'(\Delta))$. So for an arbitrary Γ and Δ, we have a one-hole derivation, which we proceed by case analysis upon D'.

The two-dimensional notation kind of fails us here because of the higher-order-ness of the representational functions. In (R7) we give an alternate style of proof where instead of the generic Holey rule we transform the single Holey rule into three rules that correspond to the three cases we give here. However, while this version has more awkward notation, it's more pleasing conceptually: if we have one-hole contexts as linear functions on the object level, it is worth thinking about having them on the meta level as well.

Case 1:

$\Gamma_1' = \bar{\lambda}x.x$

$\Delta' = \bar{\lambda}x.x,$

$D' = \lambda\Gamma.\lambda\Delta.\bar{\lambda}D.\mathrm{Holey}\ D :: \Pi\Gamma.\ \Pi\Delta.\ (\Gamma \vdash \Delta) \multimap (\Gamma \vdash \Delta)$

$E :: \Gamma'(\Gamma) \vdash \gamma$

Case 2:

$\Gamma_1' = \bar{\lambda}x.\ \Gamma_1'(x),\Gamma_2$

$\Delta' = \bar{\lambda}x.\ \Delta_1'(x),\Delta_2$

$D' = \lambda\Gamma.\lambda\Delta.\bar{\lambda}D.\mathrm{Holey}(\mathrm{Split}\ (D_1'(D))\ D_2)$

$\quad :: \Pi\Gamma.\Pi\Delta.\ (\Gamma \vdash \Delta) \multimap (\Gamma_1'(\Gamma),\Gamma_2 \vdash \Delta_1'(\Delta),\Delta_2)$

$D_1 :: \Pi\Gamma.\Pi\Delta.\ (\Gamma \vdash \Delta) \multimap (\Gamma_1'(\Gamma) \vdash \Delta_1'(\Delta))$

$D_2 :: \Gamma_2 \vdash \Delta_2$

$E :: \Gamma'(\Delta_1'(\Gamma),\Delta_2) \vdash \gamma$

To show: $\Gamma'(\Gamma_1'(\Gamma),\Gamma_2) \vdash \gamma$.

Because $\Gamma'(\Delta_1'(\Gamma),\Delta_2) = (\bar{\lambda}x.\Gamma'(\Delta_1'(\Gamma),x))(\Delta_2)$, by (R1) on D_2 and E we get

$\quad F_1 :: \Gamma'(\Delta_1'(\Gamma),\Gamma_2) \vdash \gamma$

Because $\Gamma'(\Delta_1'(\Gamma),\Gamma_2) = (\bar{\lambda}x.\Gamma'(x,\Gamma_2))(\Delta_1'(\Gamma))$, by (R2) on (Holey D_1) and F_1 we get

$\quad F_2 :: \Gamma'(\Gamma_1'(\Gamma),\Gamma_2) \vdash \gamma$

```
Case 3:
 Γ₁’ = λ̆x. Γ₁,Γ₂’(x)
 Δ’  = λ̆x. Δ₁,Δ₂’(x)
 D’  = λΓ.λΔ.λ̆D.Holey(Split D₁ (D₂’(D)))
    :: ΠΓ.ΠΔ. (Γ ⊢ Δ) -o (Γ₁,Γ₂’(Γ) ⊢ Δ₁,Δ₂’(Δ))
 D₁ :: Γ₁ ⊢ Δ₁
 D₂ :: ΠΓ.ΠΔ. (Γ ⊢ Δ) -o (Γ₂’(Γ) ⊢ Δ₂’(Δ))
 E  :: Γ’(Δ₁,Δ₂’(Γ)) ⊢ γ

 To show: Γ’(Γ₁’(Γ),Γ₂) ⊢ γ.
 Because Γ’(Δ₁,Δ₂’(Γ)) = (λ̆x.Γ’(x,Δ₂’(Γ)))(Δ₁), by (R1) on D₁ and E we get
   F₁ :: Γ’(Γ₁,Δ₂’(Γ)) ⊢ γ
 Because Γ’(Γ₁,Δ₂’(Γ)) = (λ̆x.Γ’(Γ₁,x))(Δ₂’(Γ)), by (R2) on (Holey D₂) and F₁ we get
   F₂ :: Γ’(Γ₁,Γ₂’(Γ)) ⊢ γ

%{ === Proof of (R3) === }%

Proof proceeds by case analysis on the second derivation. The fact that case 1 and 2 are
complete case analysis is a property of linear one-hole-contextey functions.

 If Γ ⊢ A⁻          and Γ’(hyp A⁻) ⊢ γ   then Γ’(Γ) ⊢ γ

Case 1:
 D :: Γ ⊢ A⁻

 E₁ :: Γ’(hyp A⁻) ≡ Γ’(hyp A⁻)
 E₂ :: Γ’ ⊢ [A⁻] > γ
 ------------------------------------------- Enter Left Focus
 Γ’(hyp A⁻) ⊢ γ

 To show: Γ’(Γ) ⊢ γ. By (-) on D and E₂

Case 2:
 D :: Γ ⊢ A⁻

 E₁ :: Γ’(hyp A⁻) ≡ Γ’’(hyp A⁻)(hyp B⁻)
 E₂ :: Γ’’(hyp A) ⊢ [B⁻] > γ
 ------------------------------------------- Enter Left Focus
 Γ’(hyp A⁻) ⊢ γ

 To show: Γ’’(Γ)(hyp B⁻) ⊢ γ.
 By (R4) on D and E₂, F₂ :: Γ’’(Γ) ⊢ [B⁻] > γ
 We have F₁ :: Γ’(Γ) ≡ Γ’’(Γ)(hyp B⁻)
 By rule on F₁ and F₂, F :: Γ’(Γ) ⊢ γ

Case 3:
 D :: Γ ⊢ A⁻

 E :: Γ’(hyp A⁻) ⊢ [B⁺]
 ------------------------------------------- Enter Right Focus
 Γ’(hyp A⁻) ⊢ B⁺

 To show: Γ’(Γ) ⊢ B⁺.
 By (R5) on D and D and E, F₁ :: Γ’(Γ) ⊢ [B⁺]
 By rule on F₁, F :: Γ’(Γ) ⊢ B⁺
```

Proof of (R4)

Case analysis on the second derivation (essentially it's just a case analysis on which branch the principal formula ends up in).

Case 1:
$D :: \Gamma \vdash A^-$

$E_1 :: \Delta' \Vdash B^- > \gamma_0$
$E_2 :: \Gamma''(hyp\ A^-) \equiv \Gamma o''(hyp\ A^-) \circ \Gamma i'$
$E_3 :: \Gamma i' \vdash \Delta'$
$E_4 :: \Gamma o''(hyp\ A^-) \vdash \gamma_0 > \gamma$
-- Perform Left Focus
$\Gamma''(hyp\ A^-) \vdash [B^-] > \gamma$

To show: $\Gamma''(hyp\ A^-) \vdash [A^-] > \gamma$
We have $F_2 :: \Gamma''(\Gamma) \equiv \Gamma o''(\Gamma) \circ \Gamma i'$
By (R6) on D and E_4, $F_4 :: \Gamma o''(hyp\ A^-) \vdash \gamma_0 > \gamma$
By rule on E_1, F_2, E_3, F_4, $F :: \Gamma''(hyp\ A^-) \vdash [B^-] > \gamma$

Case 2:
$D :: \Gamma \vdash A^-$

$E_1 :: \Delta' \Vdash B^- > \gamma_0$
$E_2 :: \Gamma''(hyp\ A^-) \equiv \Gamma o' \circ \Gamma i''(hyp\ A^-)$
$E_3 :: \Gamma i''(hyp\ A^-) \vdash \Delta'$
$E_4 :: \Gamma o' \vdash \gamma_0 > \gamma$
-- Perform Left Focus
$\Gamma''(hyp\ A^-) \vdash [B^-] > \gamma$

To show: $\Gamma'' \vdash [B^-] > \gamma$
We have $F_2 :: \Gamma''(\Gamma) \equiv \Gamma o' \circ \Gamma i''(\Gamma)$
By (R7) on D and E_3, $F_3 :: \Gamma i''(\Gamma) \vdash \Delta'$
By rule on E_1, F_2, F_3, E_4, $F :: \Gamma''(\Gamma) \vdash [B^-] > \gamma$

Proof of (R5)

$D :: \Gamma \vdash A^-$

$E_1 :: \Delta \Vdash B^+$
$E_2 :: \Gamma'(hyp\ A^-) \vdash \Delta$
-- Perform Right Focus
$\Gamma'(hyp\ A^-) \vdash [B^+]$

To show: $\Gamma'(\Gamma) \vdash [B^+]$
By (R8) on D and E_2, $F_2 :: \Gamma'(\Gamma) \vdash \Delta$
By rule on E_1, F_2, $F :: \Gamma'(\Gamma) \vdash [B^+]$

Proof of (R6)

Case analysis on the second derivation.

Case 1:

$D :: \Gamma \vdash A$

$E :: \Delta \Vdash B^{+} \;\Rightarrow\; \Gamma''(\text{hyp } A^{-})(\Delta) \vdash \gamma$

--- Left inversion

$\Gamma''(\text{hyp } A) \vdash B^{+} > \gamma$

To show: $\Gamma''(\Gamma) \vdash B^{+} > \gamma$

I need to create a computational function from $\Delta \Vdash B^{+}$ to $\Gamma''(\Gamma)(\Delta) \vdash \gamma$.

Assume $F_1 :: \Delta \Vdash B^{+}$

By application of F_1 to E, $E_2 :: \Gamma''(\text{hyp } A^{-})(\Delta) \vdash \gamma$

By (R3) on D and E_2, $F_2 :: \Gamma''(\Gamma)(\Delta) \vdash \gamma$

Therefore, $F :: \Delta \Vdash B^{+} \;\Rightarrow\; \Gamma''(\Gamma)(\Delta) \vdash \gamma$

By rule, $\Gamma''(\Gamma) \vdash B^{+} > \gamma$

We have to be careful in justifying the application of induction hypothesis (R3) in Case 1. However, because this is an iterated inductive definition, we are justified in calling E_2 a smaller derivation.

Case 2:

$D :: \Gamma \vdash A$

$E :: \Gamma''(\text{hyp } A^{-}) \equiv []$

--- Atom

$\Gamma''(\text{hyp } A^{-}) \vdash Q^{-} > Q^{-}$

Immediate from the contradiction implied by E.

Proof of (R7)

Case analysis on the second derivation. Remember, as we said when proving (R2), here we use the derived rules for the Holey derivation. The first analogue would be making one-hole contexts a formal structure $\Sigma ::= [] \mid \Sigma,\Gamma \mid \Gamma,\Sigma$

Case 1:

$D :: \Gamma \vdash A$

$E :: \Gamma''(\text{hyp } A^{-}) \equiv []$

--- Holey/Hole

$\Gamma''(\text{hyp } A^{-}) \vdash []$

Immediate from the contradiction implied by E.

Case 2:

$D :: \Gamma \vdash A$

$E_1 :: \Gamma_1''(\text{hyp } A^{-}) \vdash \Psi_1'$

$E_2 :: \Gamma_2 \vdash \Psi_2$

--- Holey/Left

$\Gamma_1''(\text{hyp } A^{-}),\Gamma_2 \vdash \Psi_1',\Psi_2$

By (R7) on D and E_1, $F_1 :: \Gamma_1''(\Gamma) \vdash \Psi_1'$

By rule on F_1 and E_2, $\Gamma_1''(\Gamma),\Gamma_2 \vdash \Psi_1',\Psi_2$

Case 3:
$D :: \Gamma \vdash A$

$E_1 :: \Gamma_1' \vdash \Psi_1'$
$E_2 :: \Gamma_2'(\text{hyp } A^-) \vdash \Psi_2$
-- Holey/Left
$\Gamma_1', \Gamma_2'(\text{hyp } A^-) \vdash \Psi_1', \Psi_2$

By (R8) on D and E_2, $F_2 :: \Gamma_2'(\Gamma) \vdash \Psi_2$
By rule on E_1 and F_2, $\Gamma_1', \Gamma_2'(\Gamma) \vdash \Psi_1', \Psi_2$

Case 4:
$D :: \Gamma \vdash A$

$E_1 :: \Gamma_1'(\text{hyp } A^-) \vdash \Psi_1$
$E_2 :: \Gamma_2' \vdash \Psi_2'$
-- Holey/Right
$\Gamma_1'(\text{hyp } A^-), \Gamma_2' \vdash \Psi_1, \Psi_2'$

By (R8) on D and E_1, $F_1 :: \Gamma_1'(\Gamma) \vdash \Psi_1$
By rule on F_1 and E_2, $\Gamma_1'(\Gamma), \Gamma_2' \vdash \Psi_1, \Psi_2'$

Case 5:
$D :: \Gamma \vdash A$

$E_1 :: \Gamma_1 \vdash \Psi_1$
$E_2 :: \Gamma_2''(\text{hyp } A^-) \vdash \Psi_2'$
-- Holey/Right
$\Gamma_1, \Gamma_2''(\text{hyp } A^-) \vdash \Psi_1, \Psi_2'$

By (R7) on D and E_2, $F_2 :: \Gamma_2''(\Gamma) \vdash \Psi_2'$
By rule on E_1 and F_2, $\Gamma_1, \Gamma_2''(\Gamma) \vdash \Psi_1, \Psi_2'$

Proof of (R8)

Proof (as pretty much always) is by case analysis on the second derivation. The rules Unit and Atom contradict the assumption that our context takes the form $\Gamma'(\text{hyp } A^-)$, so we will not consider them.

Case 1:
$D :: \Gamma \vdash A$

$E_1 :: \Gamma_1'(\text{hyp } A^-) \vdash \Psi_1$
$E_2 :: \Gamma_2 \vdash \Psi_2$
-- Split
$\Gamma_1'(\text{hyp } A^-), \Gamma_2 \vdash \Psi_1, \Psi_2$

By (R8) on D and E_1, $F_1 :: \Gamma_1'(\Gamma) \vdash \Psi_1$
By rule on F_1 and E_2, $\Gamma_1'(\Gamma), \Gamma_2 \vdash \Psi_1, \Psi_2$

Case 2:
$D :: \Gamma \vdash A$

$E_1 :: \Gamma_1 \vdash \Delta_1$
$E_2 :: \Gamma_2'(\text{hyp } A^-) \vdash \Delta_2$
-- Split
$\Gamma_1, \Gamma_2'(\text{hyp } A^-) \vdash \Delta_1, \Delta_2$

By (R8) on D and E_1, $F_1 :: \Gamma_1'(\Gamma) \vdash \Psi_1$
By rule on F_1 and E_2, $\Gamma_1'(\Gamma), \Gamma_2 \vdash \Psi_1, \Psi_2$

Case 3:

$D :: \Gamma \vdash A$

$E :: \Gamma'(\text{hyp } A^-) \vdash B^-$
-- Invert and prove
$\Gamma'(\text{hyp } A^-) \vdash \text{hyp } B^-$

By (R9) on D and E, $F :: \Gamma'(\Gamma) \vdash B^-$
By rule on F, $\Gamma'(\Gamma) \vdash \text{hyp } B^-$

Proof of (R9)

$D :: \Gamma \vdash A^-$

$E :: \Delta' \Vdash B^- > \gamma \;\Rightarrow\; \Delta'(\Gamma'(\text{hyp } A^-)) \vdash \gamma$
-- Right inversion
$\Gamma'(\text{hyp } A^-) \vdash B^-$

I need to create a computational function from $\Delta \Vdash B^- > \gamma$ to $\Delta'(\Gamma'(\Delta)) \vdash \gamma$.
Assume $F_1 :: \Delta \Vdash B^- > \gamma$
By application of F_1 to E, $E_2 :: \Delta'(\Gamma'(\text{hyp } A^-)) \vdash \gamma$
By (R3) on D and E_2, $F_2 :: \Delta'(\Gamma'(\Gamma)) \vdash \gamma$
Therefore, $F_1 :: \Delta' \Vdash B^- > \gamma \;\Rightarrow\; \Delta'(\Gamma'(\Gamma)) \vdash \gamma$
By rule on E, $F :: \Gamma'(\Gamma) \vdash B^-$

Proof of (L1)

Case analysis on the first derivation:

Case 1:

$D_1 :: \Gamma \vdash [A^+]$
-- Enter Right Focus
$\Gamma \vdash A^+$

$E :: \Gamma' \vdash A^+ > \gamma$

By (+) on D_1 and E, $F_1 :: \Gamma'(\Gamma) \vdash \gamma$

Case 2:

$D_1 :: \Gamma_1' \vdash [B^-] > \gamma_0$
-- Enter Left Focus
$\Gamma_1'(\text{hyp } B^-) \vdash \gamma_0$

$E :: \Gamma' \vdash \gamma_0 > \gamma$

By (L2) on D_1 and E, $F_1 :: \Gamma' \circ \Gamma_1' \vdash [B^-] > \gamma$
By rule, $F :: \Gamma'(\Gamma_1'(\text{hyp } B^-)) \vdash \gamma$

Proof of (L2)

$D_1 :: \Delta' \Vdash A^- > \gamma_1$
$D_2 :: \Gamma_1' \equiv \Gamma o' \circ \Gamma i'$
$D_3 :: \Gamma i' \vdash \Delta'$
$D_4 :: \Gamma o' \vdash \gamma_1 > \gamma_0$
--- Perform Left Focus
$\Gamma_1' \vdash [A^-] > \gamma_0$

$E :: \Gamma' \vdash \gamma_0 > \gamma$

By (L3) on D_4 and E, $F_4 :: \Gamma' \circ \Gamma o' \vdash \gamma_1 > \gamma$
We have $F_2 :: \Gamma' \circ \Gamma_1' = (\Gamma' \circ \Gamma o') \circ \Gamma i'$
By rule on D_1, F_2, D_3, and D_4, $F :: \Gamma' \circ \Gamma_1' \vdash \gamma_1 > \gamma$

Proof of (L3)

Case analysis on the first derivation.

Case 1:
$D_1 :: \Delta \Vdash A^+ \Rightarrow \Gamma_1'(\Delta) \vdash \gamma_0$
--- Left inversion
$\Gamma_1' \vdash A^+ > \gamma_0$

$E :: \Gamma' \vdash \gamma_0 > \gamma$

I need to create a computational function from $\Delta \Vdash A^+$ to $\Gamma' \circ \Gamma_1' \vdash \gamma$.
Assume $F_0 :: \Delta \Vdash A^+$
By application of F_0 to D_1, $D_2 :: \Gamma_1'(\Delta) \vdash \gamma_0$
By (L1) on D_2 and E, $F_2 :: \Gamma'(\Gamma_1'(\Delta)) \vdash \gamma$
Therefore, $F_1 :: \Delta \Vdash A^+ \Rightarrow \Gamma' \circ \Gamma_1' \vdash \gamma$
By rule, $\Gamma' \circ \Gamma_1' \vdash A^+ > \gamma$

Case 2:
-------------------------- Atom
$[] \vdash Q^- > Q^-$

$E :: \Gamma' \vdash Q^- > \gamma$

To show: $\Gamma' \vdash Q^- > \gamma$. Immediate from E