Wavelets

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Function Representations

• sequence of samples (time domain)
  – finite difference method
• pyramid (hierarchical)
• polynomial
• sinusoids of various frequency (frequency domain)
  – Fourier series
• piecewise polynomials (finite support)
  – finite element method, splines
• wavelet (hierarchical, finite support)
  – (time/frequency domain)
What Are Wavelets?

In general, a family of representations using:
• hierarchical (nested) basis functions
• finite ("compact") support
• basis functions often orthogonal
• fast transforms, often linear-time
Simple Example: Haar Wavelet

- Consider piecewise-constant functions over $2^j$ equal sub-intervals of $[0,1]$

- $j=2$: four intervals

- A basis
Nested Function Spaces for Haar Basis

- Let $V_j$ denote the space of all piecewise-constant functions represented over $2^j$ equal sub-intervals of $[0,1]$

- $V_j$ has basis
Function Representations – Desirable Properties

- generality – approximate anything well
  - discontinuities, nonperiodicity, ...
- adaptable to application
  - audio, pictures, flow field, terrain data, ...
- compact – approximate function with few coefficients
  - facilitates compression, storage, transmission
- fast to compute with
  - differential/integral operators are sparse in this basis
  - Convert $n$-sample function to representation in $O(n \log n)$ or $O(n)$ time
Wavelet History, Part 1

• 1805 Fourier analysis developed
• 1965 Fast Fourier Transform (FFT) algorithm

…

• 1980’s beginnings of wavelets in physics, vision, speech processing (ad hoc)
• … little theory … why/when do wavelets work?
• 1986 Mallat unified the above work
• 1985 Morlet & Grossman continuous wavelet transform … asking: how can you get perfect reconstruction without redundancy?
Wavelet History, Part 2

• 1985 Meyer tried to prove that no orthogonal wavelet other than Haar exists, found one by trial and error!
• 1987 Mallat developed multiresolution theory, DWT, wavelet construction techniques (but still noncompact)
• 1988 Daubechies added theory: found compact, orthogonal wavelets with arbitrary number of vanishing moments!
• 1990’s: wavelets took off, attracting both theoreticians and engineers
Time-Frequency Analysis

- For many applications, you want to analyze a function in both time and frequency.
- Analogous to a musical score.

- Fourier transforms give you frequency information, smearing time.
- Samples of a function give you temporal information, smearing frequency.

- Note: substitute “space” for “time” for pictures.
Comparison to Fourier Analysis

• Fourier analysis
  – Basis is global
  – Sinusoids with frequencies in arithmetic progression

• Short-time Fourier Transform (& Gabor filters)
  – Basis is local
  – Sinusoid times Gaussian
  – Fixed-width Gaussian “window”

• Wavelet
  – Basis is local
  – Frequencies in geometric progression
  – Basis has constant shape independent of scale
Wavelet Applications

• Medical imaging
• Pictures less corrupted by patient motion than with Fourier methods
• Astrophysics
• Analyze clumping of galaxies to analyze structure at various scales, determine past & future of universe
• Analyze fractals, chaos, turbulence
Wavelets for Denoising

• White noise is independent random fluctuations at each sample of a function
• White noise distributes itself uniformly across all coefficients of a wavelet transform
• Wavelet transforms tend to concentrate most of the “energy” in a small number of coefficients

• Throw out the small coefficients and you’ve removed (most of) the noise
• Little knowledge about noise character required!
Wavelets, Vision, and Hearing

Human vision & hearing:
- The retina and brain have receptive fields (filters) sensitive to spots and edges at a variety of scales and translations
- Somewhat similar to wavelet multiresolution analysis
- Human hearing also uses approximately constant shape filters

Computer vision:
- Pyramid techniques popular and powerful for matching, tracking, recognition
- Gaussian & Laplacian pyramids – wavelet precursors

Speech processing:
- Quadrature Mirror Filters – wavelet precursors