$\mathrm{SL}_3(\mathbb{F}_p[X])$ has property (T): a re-presentation

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Abstract

We present a non-novel proof of Kazhdan's theorem [Kaž67] that $\mathrm{SL}_3(\mathbb{F}_p[X])$ has property (T) for $p \geq 5$. The presentation follows the methods of Dymara–Januszkiewicz [DJ02], Ershov–Jaikin-Zapirain [EJZ10], Kassabov [Kas11], Oppenheim [Opp17], and Caprace–Conder–Kaluba–Witzel [CCKW22]; it is also inspired by Kaufman–Oppenheim [KO23], Harsha–Saptharishi [HS24], and Grave de Peralta–Valentiner-Branth [GdPVB25]. The goal is to show a use of Ozawa's SOS criterion, and to make the overall argument slightly more succinct.

1 Property (T) proof

We start with a general theorem:

Theorem 1. *Let G be a group generated by subgroups G_1, G_2, G_3 of equal order, and assume that $G_{ij} \coloneqq \langle G_i, G_j \rangle$ is finite for all $i \neq j \in \{1,2,3\}$. Assume each Cayley graph $\mathrm{Cay}(G_{ij}, G_i \sqcup G_j)$ has (normalized) spectral gap exceeding $\frac{1}{4}$. Then G has property (T).*

Proof. In the group algebra $\mathbb{R}[G]$, write

$$P_i = \operatorname*{avg}_{g \in G_i}\{g\}, \qquad \Delta_i = 1 - P_i, \qquad \Delta = \operatorname*{avg}_{i \in [3]}\{\Delta_i\}. \tag{1}$$

Here Δ is the (normalized) Laplacian element with respect to the generating multiset $S = G_1 \sqcup G_2 \sqcup G_3$. Per Ozawa [Oza16], to show G has property (T), it is necessary and sufficient to show

$$\Delta^2 - \kappa\Delta \text{ is SOS} \tag{2}$$

for some $\kappa > 0$. Here by "SOS" we mean a sum of (Hermitian) squares in $\mathbb{R}[G]$. Using $P_i^2 = P_i$, it is easy to check that

$$\Delta^2 - \kappa\Delta = \frac{4}{3}\operatorname*{avg}_{i \neq j}\left\{\Delta_{ij}^2 - \left(\frac{1}{4} + \frac{3}{4}\kappa\right)\Delta_{ij}\right\}, \tag{3}$$

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where $\Delta_{ij} \coloneqq \operatorname{avg}\{\Delta_i, \Delta_j\}$. Thus it suffices to show each

$$\zeta_{ij} \coloneqq \Delta_{ij}^2 - \left(\frac{1}{4} + \frac{3}{4}\kappa\right)\Delta_{ij} \text{ is SOS} \tag{4}$$

Since ζ_{ij} is in the finite-dimensional subalgebra $\mathbb{R}[G_{ij}]$, it is SOS iff its regular representation is PSD; equivalently, iff the (normalized) Laplacian L_{ij} of the Cayley graph $\operatorname{Cay}(G_{ij}, G_i \sqcup G_j)$ satisfies

$$L_{ij}^2 \gtrsim \left(\frac{1}{4} + \frac{3}{4}\kappa\right)L_{ij}. \tag{5}$$

But by hypothesis, we can select $\kappa > 0$ such that this holds for all $i \neq j$. □

Remark 2. It's not essential for G_1, G_2, G_3 to have equal order; it just lets us avoid introducing weightings in the Cayley graphs.

We can instantiate the preceding theorem as follows:

Corollary 3. *For $p > 4$ prime, let G be the subgroup of $\mathrm{SL}_3(\mathbb{F}_p[t])$ consisting of matrices that are upper-unitriangular mod (t). Then G has property (T).*

Proof. Within $\mathrm{SL}_3(\mathbb{F}_p[t])$, define the order-$p$ subgroups

$$G_1 = \langle e_{12}(1)\rangle, \qquad G_2 = \langle e_{23}(1)\rangle, \qquad G_3 = \langle e_{31}(t)\rangle, \tag{6}$$

where as usual $e_{ij}(r)$ denote the elementary matrix with 1's on the diagonal, r in the (i, j) position, and 0's elsewhere. Then it is not too hard to show that $G = \langle G_1, G_2, G_3\rangle$. The group G_{12} is the discrete Heisenberg group modulo p; moreover $G_{23}, G_{31} \cong G_{12}$ and the relevant Cayley graphs are isomorphic. Thus it suffices to show $\operatorname{Cay}(G_{12}, G_1 \sqcup G_2)$ has spectral gap exceeding $\frac{1}{4}$. This particular Cayley graph is well-studied and it is known (see Proposition 5 below) that its normalized Laplacian has eigenvalues $\frac{1}{2} \pm \frac{1}{2\sqrt{p}}$ and $0, \frac{1}{2}, 1$. We have $\frac{1}{2} - \frac{1}{2\sqrt{p}} > \frac{1}{4}$ since $p > 4$, completing the proof. □

One can either be satisfied to have property (T) for this G (which is almost $\mathrm{SL}_3(\mathbb{F}_p[t])$); or, one can work with $G_3 = \langle e_{31}(1), e_{31}(t)\rangle$ (but this requires a generalization of Proposition 5 below); or, one can quote literature to get:

Corollary 4. *For $p > 4$ prime, $\mathrm{SL}_3(\mathbb{F}_p[t])$ has property (T).*

Proof. Since G from Corollary 3 is a finite-index subgroup of the discrete group $\mathrm{SL}_3(\mathbb{F}_p[t])$, we can appeal to the basic fact [BdlHV08, Thm. 1.7.1] that property (T) for G implies property (T) for $\mathrm{SL}_3(\mathbb{F}_p[t])$. □

2 Spectrum of Heisenberg Cayley graphs

Regarding the spectrum of the Heisenberg Cayley graphs, if one just wants a particular group with property (T), one can fix $p = 5$; then, verifying the spectral gap exceeds $\frac{1}{4}$ only requires computing the eigenvalues of an explicit 125-vertex graph. More generally, we have the following (known) fact:

Proposition 5. *Let U be the group of upper-unitriangular 3×3 matrices mod prime p. Define subgroups*

$$H_1 = \left\{ \begin{pmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : 0 \le s < p \right\}, \qquad H_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} : 0 \le t < p \right\}. \tag{7}$$

Then the Cayley graph $C = \mathrm{Cay}(U, H_1 \sqcup H_2)$ has (normalized) spectral gap $\frac{1}{2} - \frac{1}{2\sqrt{p}}$.

Proof. Let B be the following (connected, p-regular) bipartite point-line incidence graph for $\mathbb{F}_p^2$: The left vertices of B are all points $(x, y) \in \mathbb{F}_p^2$. The right vertices are all (non-vertical) lines with slope/intercept pair $(m, b) \in \mathbb{F}_p^2$. Edges in B represent point-line incidences, meaning $y = mx + b$. We can also uniquely associate each such edge with an element

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} \in U. \tag{8}$$

Let M be the edge-vertex incidence matrix of B, so each row has exactly two 1's. We investigate the matrices $MM^\top$ and $M^\top M$ (which have the same eigenvalues, excluding 0's).

On one hand, $MM^\top$ is the adjacency matrix of the edge-vertex-edge walk. This has two types of steps; those that pass through point-vertices and those that pass through-line vertices. The former involves fixing the point (x, y) and choosing a new slope m'; the latter involves fixing the line (m, b) and choosing a new point (x', y') on the line (with $y' - mx' = y - mx$). But these steps are precisely achieved by

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & x & y \\ 0 & 1 & m+t \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & x & y \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & x+s & y+sm \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix}, \tag{9}$$

for $0 \le s, t < p$; in other words, left-multiplication by elements of H_2, H_1 (respectively). Thus $MM^\top$ is the (unnormalized) adjacency matrix of C.

On the other hand, $M^\top M$, the adjacency matrix for the vertex-edge-vertex walk in B, has

$$M^\top M = \begin{pmatrix} pI & N \\ N^\top & pI \end{pmatrix}, \tag{10}$$

where N is the point-line incidence matrix. Thus $N^\top N$ is the adjacency matrix for the line-point-line walk. This graph has p self-loops at each line, and otherwise looks like the complete p-partite graph, where the parts are the slopes. (This is because two lines do not intersect if they are parallel; otherwise they intersect uniquely.) Hence

$$N^\top N = pI \otimes I + J \otimes J - I \otimes J, \tag{11}$$

where I is the $p \times p$ identity matrix and J is the $p \times p$ matrix of 1's. Since I's eigenvalues are $1, \dots, 1$ and J's are $p, 0, \dots, 0$, we conclude the (distinct) eigenvalues of $N^\top N$ are $0, p, p^2$. It then follows from Equation (10) that the eigenvalues of $M^\top M$ are $p, p \pm \sqrt{p}, p \pm p$.

Thus these are also the eigenvalues of $MM^\top$; in other words, the normalized eigenvalues of C are $0, \frac{1}{2} \pm \frac{1}{2\sqrt{p}}, \frac{1}{2}, 1$. Since C is connected, the second-largest normalized eigenvalue is $\frac{1}{2} + \frac{1}{2\sqrt{p}}$, hence the gap is as claimed. □

3 A strongly explicit 6-regular expander family

Based on the preceding results, one can give an elementary proof of the following:

Theorem 6. *For n a power of 2, consider the following 6-regular graph G_n on $N = 5^{3n} - 1$ vertices: The vertex set consists of triples (x, y, z) for $x, y, z \in \mathbb{F}_5^n$ (not all zero). Each (x, y, z) is connected to $(x \pm y, y, z)$ and $(x, y \pm z, z)$ and $(x, y, z \pm \mathrm{shift}(x))$, where $\mathrm{shift}(x_0, x_1, x_2, \ldots, x_{n-1}) = (2x_{n-1}, x_0, x_1, \ldots, x_{n-2})$. Then G_n is connected, and its adjacency matrix has second-largest eigenvalue at most $\frac{5+3\sqrt{5}}{2} < 5.86$.*

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