1 The rationality of weakly symmetric social choice functions

Lemma 1.1 Let $f : \{-1, 1\}^n \to \mathbb{R}$ be weakly symmetric (transitive). Then $\hat{f}(i)$ is the same for all $i \in [n]$.

Proof: Let $i \neq i'$. Since $f$ is weakly symmetric, we can pick a permutation $\pi$ on $[n]$ such that $\pi(i) = i'$ and

$$f(x_{\pi(1)}, \ldots, x_{\pi(n)}) = f(x_1, \ldots, x_n)$$

for all $x \in \{-1, 1\}^n$. Now

$$\hat{f}(i) = \mathbb{E}[f(x)x_i] = \mathbb{E}[f(x_{\pi(1)}, \ldots, x_{\pi(n)})x_{\pi(i)}],$$

since a uniformly random string hit by a permutation is still uniformly random. But the above equals $\mathbb{E}[f(x)x_{i'}] = \hat{f}(i')$. □

Proposition 1.2 (Kalai 2002) If $f : \{-1, 1\}^n \to \{-1, 1\}$ is a weakly symmetric social choice function, $\text{Rationality}(f) \leq \frac{7}{9} + \frac{4}{9\pi} + O(1/n) \approx .919$.

Proof: Since $f$ is weakly symmetric, the lemma implies that $\hat{f}(i)$ is the same for all $i$’s. Hence

$$W_1(f) = \sum_{i=1}^n \hat{f}(i)^2 = \frac{1}{n} \left( \sum_{i=1}^n \hat{f}(i) \right)^2.$$

But we proved that for any $f$, $\sum_{i=1}^n \hat{f}(i)$ is at most that of Majority, namely $\sqrt{2/\pi} \sqrt{n} + O(1/\sqrt{n})$. Hence

$$W_1(f) \leq \frac{2}{\pi} + O(1/n),$$

and the result follows from our proposition bounding $\text{Rationality}(f)$ in terms of $W_1(f)$. □