8803 Connections between Learning, Game Theory, and Optimization

Homework # 2 **Due: October 28th 2010**

This homework is due by the start of class on October 28th 2010. You can either submit the homework via the course page on T-Square or hand it in at the beginning of the class on October 28th 2010. Start early!

Groundrules:

- Your work will be graded on correctness, clarity, and conciseness.
- You may collaborate with others on this problem set. However, you must *write your own solutions* and *list your collaborators/sources* for each problem.

Problems:

1. **VC-dimension of linear separators:** In this problem you will prove that the VC-dimension of the class H_n of halfspaces (another term for linear threshold functions) in n dimensions is $n+1$. We will use the following definition: The *convex hull* of a set of points S is the set of all convex combinations of points in S; this is the set of all points that can be written as $\sum_{x_i \in S} \lambda_i x_i$, where each $\lambda_i \geq 0$, and $\sum_i \lambda_i = 1$. It is not hard to see that if a halfspace has all points from a set S on one side, then the entire convex hull of S must be on that side as well.

   (a) **[lower bound]** Prove that VC-dim$(H_n) \geq n+1$ by presenting a set of $n+1$ points in n-dimensional space such that one can partition that set with halfspaces in all possible ways. (And, show how one can partition the set in any desired way.)

   (b) **[upper bound part 1]** The following is "Radon's Theorem," from the 1920's.

   Theorem. *Let S be a set of $n+2$ points in n dimensions. Then S can be partitioned into two (disjoint) subsets S_1 and S_2 whose convex hulls intersect.*

   Show that Radon's Theorem implies that the VC-dimension of halfspaces is *at most* $n+1$. Conclude that VC-dim$(H_n) = n+1$.

   (c) **[upper bound part 2]** Now we prove Radon's Theorem. We will need the following standard fact from linear algebra. If $x_1, \ldots, x_{n+1}$ are $n+1$ points in n-dimensional space, then they are linearly dependent. That is, there exist real values $\lambda_1, \ldots, \lambda_{n+1}$ *not all zero* such that $\lambda_1 x_1 + \ldots + \lambda_{n+1} x_{n+1} = 0$.

   You may now prove Radon's Theorem however you wish. However, as a suggested first step, prove the following. For any set of $n+2$ points $x_1, \ldots, x_{n+2}$ in n-dimensional space, there exist $\lambda_1, \ldots, \lambda_{n+2}$ *not all zero* such that $\sum_i \lambda_i x_i = 0$ and $\sum_i \lambda_i = 0$. (This is called *affine dependence.*)

2. **Symmetric congestion games:** Consider a network congestion game defined on a directed graph $G = (V, E)$. Assume each player i wants to get from source o and to destination t (so all players have the same set of strategies, which is the set of paths from o to t). Assume that the cost of each edge is monotonically increasing with the number of players using that

edge. Show that these games have a pure Nash equilibrium and provide a polynomial time algorithm to compute a pure Nash equilibrium.

Hints: Try to compute the solution that minimizes the potential function. Think of min-cost network flow.

Extra Credit:

1. **Boosting and Game Theory:** Think about the boosting process in relation to the decision theoretic experts model and derive a boosting algorithm based on this connection.

   Note: You are welcome to read the paper "Game Theory, On-line Prediction and Boosting" by Rob Schapire and Yoav Freund, but please write the solution in your own words.