GRASP-an efficient SAT solver

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What is SAT?

- Given a propositional formula in CNF, find an assignment to boolean variables that makes the formula true!
- E.g.

$\omega_1 = (x_2 \vee x_3)$

$\omega_2 = (\neg x_1 \vee \neg x_4)$

$\omega_3 = (\neg x_2 \vee x_4)$

$A = \{x_1=0, x_2=1, x_3=0, x_4=1\}$

SATisfying assignment!

What is SAT?

- Solution 1: Search through all assignments!
  - n variables $\rightarrow 2^n$ possible assignments, explosion!
- SAT is a classic NP-Complete problem, solve SAT and P=NP!

Why SAT?

- Fundamental problem from theoretical point of view
- Numerous applications
  - CAD, VLSI
  - Optimization
  - Model Checking and other type of formal verification
  - AI, planning, automated deduction

Outline

- Terminology
- Basic Backtracking Search
- GRASP
- Pointers to future work

Please interrupt me if anything is not clear!

Terminology

- CNF formula φ
  - $x_1,\ldots, x_n$: n variables
  - $\omega_1,\ldots, \omega_m$: m clauses

$\omega_1 = (x_2 \vee x_3)$

$\omega_2 = (\neg x_1 \vee \neg x_4)$

$\omega_3 = (\neg x_2 \vee x_4)$

$A = \{x_1=0, x_2=1, x_3=0, x_4=1\}$

- Assignment A
  - Set of $(x,v(x))$ pairs
  - $|A| < n \rightarrow$ partial assignment $\{(x_1,0), (x_2,1), (x_4,1)\}$
  - $|A| = n \rightarrow$ complete assignment $\{(x_1,0), (x_2,1), (x_3,0), (x_4,1)\}$
  - $\varphi|_A = 0 \rightarrow$ unsatisfying assignment $\{(x_1,1), (x_4,1)\}$
  - $\varphi|_A = 1 \rightarrow$ satisfying assignment $\{(x_1,0), (x_2,1), (x_4,1)\}$

Terminology

- Assignment *A (contd.)*
  - $\varphi|_A = X \rightarrow$ unresolved *{(x_1,0), (x_2,0), (x_4,1)}*
- An assignment partitions the clause database into three classes
  - Satisfied, unsatisfied, unresolved
- Free literals: unassigned literals of a clause
- Unit clause: # free literals = 1

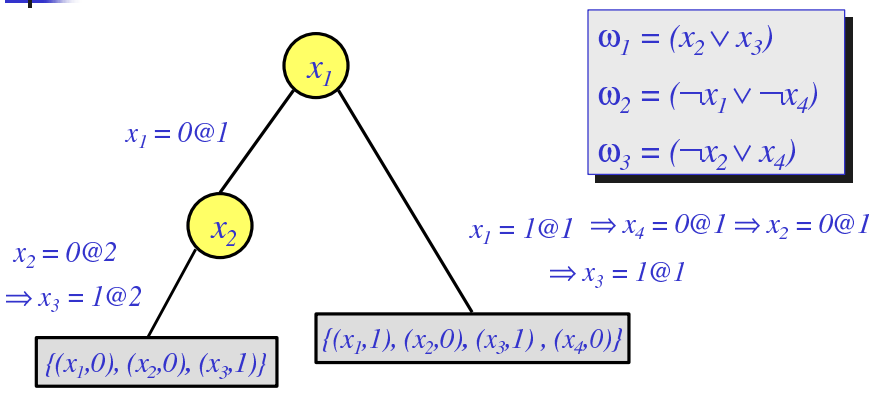
Basic Backtracking Search

- Organize the search in the form of a decision tree
  - Each node is an assignment, called decision assignment
  - Depth of the node in the decision tree → decision level $\delta(x)$
  - $x=v@d \rightarrow x$ is assigned to v at decision level d

Basic Backtracking Search

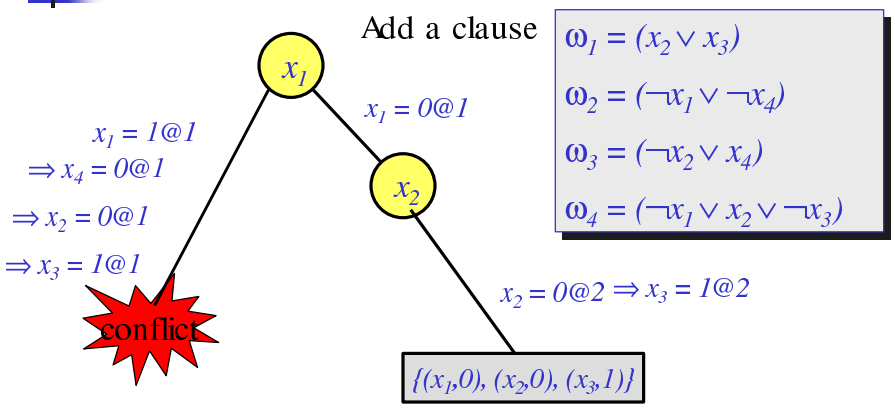
- Iterate through:
  1. Make new decision assignments to explore new regions of search space
  2. Infer implied assignments by a deduction process. May lead to unsatisfied clauses, **conflict!** The assignment is called conflicting assignment.
  3. Conflicting assignments leads to backtrack to discard the search subspace

Backtracking Search in Action



No backtrack in this example!

Backtracking Search in Action



Davis-Putnam revisited

The fastest known algorithms for deciding propositional satisfiability are based on the Davis-Putnam Algorithm.

A *unit clause* is a clause that consists of a single literal

```
function Satisfiable (clause list S) returns boolean:
    /* unit propagation */
    repeat
        for each unit clause L ∈ S do
            delete from S every clause containing L
            delete ¬L from every clause of S in which it occurs
        end for
        if S is empty then return TRUE
        else if null clause is in S then return FALSE end if
    until no further changes result end repeat
    /* splitting */
    choose a literal L occurring in S
    if Satisfiable (S ∪ {L}) then return TRUE
    else if Satisfiable (S ∪ {¬L}) then return TRUE
    else return FALSE end if
end function
```

Deduction

Decision

Backtrack

GRASP

- GRASP is Generalized seaRch Algorithm for the Satisfiability Problem (Silva, Sakallah, '96)
- Features:
  - Implication graphs for BCP and conflict analysis
  - Learning of new clauses
  - **Non-chronological** backtracking!

GRASP search template

```
// Global variables:                    Clause database
//                                      Partial variable assignment A
// Return value:                        FAILURE or SUCCESS
// Auxiliary variables:                 Backtracking decision level β
//
GRASP()
{
    return (!Search (0, β) != SUCCESS) ? FAILURE : SUCCESS);
}

// Input argument:                      Current decision level d
// Output argument:                     Backtracking decision level β
// Return value:                        CONFLICT or SUCCESS
//
Search (d, &β)
{
    if (Decide (d) == SUCCESS)
        return SUCCESS;
    while (TRUE) {
        if (Deduce (d) != CONFLICT) {
            if (Search (d + 1, β) == SUCCESS) return SUCCESS;
            else if (β != d) { Erase(); return CONFLICT; }
        }
        if (Diagnose (d, β) == CONFLICT) { Erase(); return CONFLICT; }
        Erase();
    }
}
```

GRASP Decision Heuristics

- Procedure `decide()`
- Choose the variable that satisfies the most # clauses == max occurences as unit clauses at current decision level
- Other possibilities exist

GRASP Deduction

- Boolean Constraint Propagation using implication graphs

  E.g. for the clause $\omega = (x \vee \neg y)$, if $y=1$, then we must have $x=1$
- For a variable x occuring in a clause , assignment 0 to all other literals is called antecedent assignment $A(x)$
  - E.g. for $\omega = (x \vee y \vee \neg z)$,

    $A(x) = \{(y,0), (z,1)\}$, $A(y) = \{(x,0),(z,1)\}$, $A(z) = \{(x,0), (y,0)\}$
  - Variables directly responsible for forcing the value of x
  - Antecedent assignment of a decision variable is empty

Implication Graphs

- Nodes are variable assignments $x=v(x)$ (decision or implied)
- Predecessors of x are antecedent assignments $A(x)$
  - No predecessors for decision assignments!
- Special conflict vertices have $A(\kappa)$ = assignments to vars in the unsatisfied clause
- Decision level for an implied assignment is

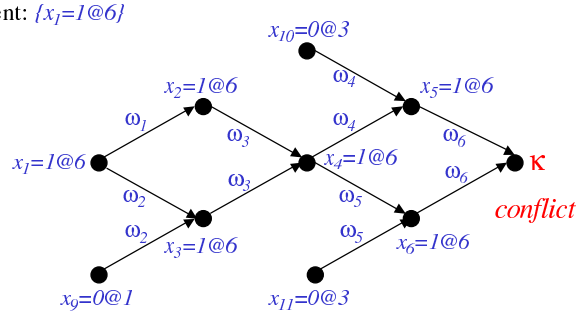
$$\delta(x) = max\{\delta(y) | (y,v(y)) \in A(x)\}$$

Example Implication Graph

Current truth assignment: $\{x_9=0@1, x_{10}=0@3, x_{11}=0@3, x_{12}=1@2, x_{13}=1@2\}$

Current decision assignment: $\{x_1=1@6\}$

$\omega_1 = (\neg x_1 \vee x_2)$
$\omega_2 = (\neg x_1 \vee x_3 \vee x_9)$
$\omega_3 = (\neg x_2 \vee \neg x_3 \vee x_4)$
$\omega_4 = (\neg x_4 \vee x_5 \vee x_{10})$
$\omega_5 = (\neg x_4 \vee x_6 \vee x_{11})$
$\omega_6 = (\neg x_5 \vee x_6)$
$\omega_7 = (x_1 \vee x_7 \vee \neg x_{12})$
$\omega_8 = (x_1 \vee x_8)$
$\omega_9 = (\neg x_7 \vee \neg x_8 \vee \neg x_{13})$



GRASP Deduction Process

```
// Global variables:              Implication graph I
// Input argument:                Current decision level d
// Return value:                  CONFLICT or SUCCESS
//
Deduce (d)
{
    while (unit clauses in φ or clauses unsatisfied) {
        if (exists unsatisfied clause ω) {
          add conflict vertex κ to I;
          record A(ω);
          return CONFLICT;
        }
        if (exists unit clause ω with free literal l = x or l = ¬x) {
          record A(ω);
          δ(x) = d;
          set x = 1 if l = x or x = 0 if l = ¬x;
        }
    }
    return SUCCESS;
}
```


GRASP Conflict Analysis

- After a conflict arises, analyze the implication graph at current decision level
- Add new clauses that would prevent the occurrence of the same conflict in the future ⇒ Learning
- Determine decision level to backtrack to, might not be the immediate one ⇒ Non-chronological backtrack

Learning

- Determine the assignment that caused the conflict, negation of this assignment is called conflict induced clause $\omega_C(\kappa)$
  - The conjunct of this assignment is necessary condition for κ
  - So adding $\omega_C(\kappa)$ will prevent the occurrence of κ again

Learning

- Find $\omega_C(\kappa)$ by a backward traversal of the IG, find the roots of the IG in the transitive fanin of κ
- For our example IG,

$$\omega_C(\kappa) = (\neg x_1 \vee x_9 \vee x_{10} \vee x_{11})$$

Learning (some math)

- For any node of an IG x, partition $A(x)$ into

$$\Lambda(x) = \{(y,v(y)) \in A(x) | \delta(y) < \delta(x)\}$$
$$\Sigma(x) = \{(y,v(y)) \in A(x) | \delta(y) = \delta(x)\}$$

- Conflicting assignment $A_C(\kappa) = causesof(\kappa)$, where

$$causesof(x) = \begin{cases} (x,v(x)) \text{ if } A(x) = \phi \\ \Lambda(x) \cup \left[\bigcup_{(y,v(y)) \in \Sigma(x)} causesof(y) \right] \text{o/w} \end{cases}$$

Learning (some math)

- Deriving conflicting clause $\omega_C(\kappa)$ from the conflicting assignment $A_C(\kappa)$ is straight forward

$$\omega_C(\kappa) = \sum_{(x,v(x)) \in A_C(\kappa)} x^{v(x)}$$

- For our IG,

$A_C(\kappa) = \{x_1=1@6, x_9=0@1, x_{10}=0@3, x_{11}=0@3\}$

Learning

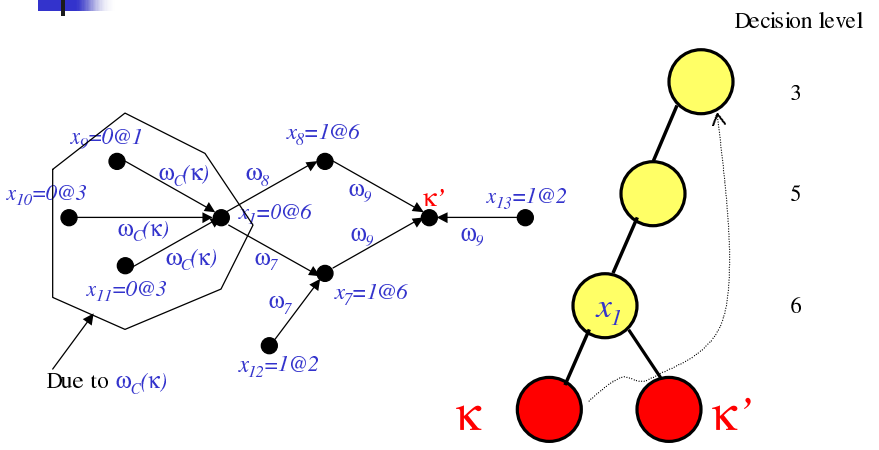
- Unique implication points (UIPs) of an IG also provide conflict clauses
- Learning of new clauses increases clause database size
- Heuristically delete clauses based on a user parameter
  - If size of learned clause > parameter, don't include it

Backtracking

Failure driven assertions (FDA):

- If $\omega_C(\kappa)$ involves current decision variable, then after addition, it becomes unit clause, so different assignment for the current variable is immediately tried.
- In our IG, after erasing the assignment at level 6, $\omega_C(\kappa)$ becomes a unit clause $\neg x_1$
- This immediately implies $x_1=0$

Continued IG



Backtracking

Conflict Directed Backtracking

- Now deriving a new IG after setting x_1=0 by FDA, we get another conflict κ'
- Non-chronological backtrack to decision level 3, because backtracking to any level 5, 4 would generate the same conflict κ'

Backtracking

Conflict Directed Backtracking contd.

$A_C(\kappa') = \{x_9=0@1, x_{10}=0@3, x_{11}=0@3, x_{12}=1@2, x_{13}=1@2\}$

$\omega_C(\kappa) = (x_9 \vee x_{10} \vee x_{11} \vee \neg x_{12} \vee \neg x_{13})$

- Backtrack level is given by

$$\beta = max\{\delta(x) | (x, v(x)) \in A_C(\kappa')\}$$

- $\beta = d\text{-}1$ chronological backtrack
- $\beta < d\text{-}1$ non-chronological backtrack

Procedure `Diagnose()`

```
// Global variables:                Implication graph I
//                                  Clause database φ
// Input variable:                  Current decision level d
// Output variable:                 Backtracking decision level β
// Return value:                    CONFLICT or SUCCESS
//
Diagnose (d, &β)
{
    ω_C(κ) = Create_Conflict_Induced_Clause();        // Using (3.4)
    Update_Clause_Database ( ω_C(κ) );
    β = Compute_Max_Level();                          // Using (3.7)
    if (β_i != d) {
        add new conflict vertex κ to I;
        record A(κ);
        return CONFLICT;
    }
    return SUCCESS;
}
```

Is that all?

- Huge overhead for constraint propagation
- Better decision heuristics
- Better learning, problem specific
- **Better engineering!**

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