GEOMETRIC DESCENT ALGORITHMS FOR ATTITUDE DETERMINATION USING GPS

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Abstract: This paper describes a set of numerical optimization algorithms for solving the GPS-based attitude determination problem. We pose the problem as one of minimizing the function $tr(\Theta N \Theta^T Q - 2\Theta W)$ with respect to $\Theta \in SO(3)$, where N, Q, and W are given 3×3 matrices. The method of steepest descent and Newton's method are generalized to $SO(3)$ by taking advantage of its Lie group structure. Analytic solutions to the line search procedure are also derived. Results of numerical experiments for the class of geometric descent algorithms proposed here are presented.

Keywords: GPS, attitude determination, optimization, steepest descent, Newton's Method, SO(n)

1. INTRODUCTION

The Global Positioning System (GPS) has now entered the mainstream of engineering, and is beginning to find applications ranging from land and air traffic control to surveying and aircraft precision landing. In this paper we address the problem of attitude determination using GPS and multiple antennas. Assuming that cycle ambiguities are already known, the problem can be posed as the minimization of a function on the rotation group $SO(3)$ of the form

$$J(\Theta) = tr(\Theta N \Theta^T Q - 2\Theta W)$$

where $\Theta \in SO(3)$, and N, Q, W are given 3×3 matrices. We show below how the usual formulation of the GPS-based attitude determination problem as given in the literature can be reduced to this particular form, and describe the physical meaning of the elements constituting the various matrices. See [6] for a more complete discussion and background on the various issues related to GPS-based attitude determination.

The GPS-based attitude determination problem can be viewed as an optimization problem on $SO(3)$, and as is well-known, $SO(3)$ does not possess a Euclidean structure. There is both a well-developed theory and an abundance of numerical algorithms for constrained minimization of functions in Euclidean space, and it is quite reasonable to apply such an algorithm to the above objective function; the set of admissible solutions in this case would be the 3×3 real matrices Θ subject to the constraints $\Theta^T \Theta = I$ and $\det(\Theta) = 1$.

In this paper we suggest an alternative approach that takes advantage of the geometric structure of the rotation group $SO(3)$, and recasts the problem as one of unconstrained minimization. More generally, in the case of compact semisimple Lie groups like $SO(3)$, it is possible to exploit the geometric and algebraic structure of the underlying space, and derive computationally efficient generalizations of unconstrained optimization algorithms that were originally intended for Euclidean space. In certain cases, and we show this to be true for our attitude determination problem, the geometric approach leads to algorithms with faster and more robust convergence properties.

Various forms of the attitude determination problem have been investigated in the aerospace literature. Closely related to our problem is the Wahba problem [16], which can be posed as the minimization of the function

$$J(\Theta) = tr(\Theta F)$$

where $F \in \Re^{3\times 3}$ is a given arbitrary nonsingular matrix constructed from sensor measurements. A complete solution to this problem based on the singular value decomposition of F can be obtained(e.g., [11], [10], [9], [7]).

Under certain simplifying assumptions the GPS-based attitude determination problem reduces to the above form, but for the general case it does not appear that a closed-form solution like that for the Wahba problem can be derived. Numerical algorithms for the solution of the Wahba problem have also been proposed in, e.g., [17], [15], [11], [1], [12], and in [13] a different optimality criterion for attitude estimation is proposed and evaluated. For the general GPS-based attitude determination problem, Cohen [6] develops a recursive algorithm for online estimation of the attitude that is based on a linearization of the objective function.

While not directed specifically at GPS-based attitude determination, in related work a number of researchers have posed optimization problems on $SO(n)$ and investigated their solutions in a geometric setting. Brockett [3] considers objective functions on $SO(n)$ of the form

$$J(\Theta) = tr(\Theta N \Theta^T Q)$$

where N, Q are given symmetric $n \times n$ matrices. He shows that a variety of continuous and combinatorial problems (e.g., the eigenvalue problem, linear programming, sorting) can be posed as the minimization of such an objective function, and derives the corresponding gradient flow equations. Chu [5], Bloch et al [2], Helmke and Moore [8], and others (see, e.g., Brockett [4] and the references cited therein) examine the geometry of these and other related optimization problems on non-Euclidean spaces.

In this article we derive a class of numerical optimization algorithms for GPS-based attitude determination that takes advantage of the special structure of $SO(3)$. Specifically, we present geometric versions of steepest descent and Newton's method applicable to our particular objective function on $SO(3)$; in both these methods the attitude is determined as the solution of an unconstrained minimization problem. We derive the detailed steps of the geometric steepest descent algorithm, including methods for finding the exact solution as well as rapid estimates of the stepsize in the line search procedure. Newton's method is then generalized to our objective function. Results of numerical experiments comparing the relative performance of these algorithms with standard minimization algorithms provided by Matlab are also given.

2. GPS-BASED ATTITUDE DETERMINATION

The standard formulation of the GPS-based attitude determination problem is to minimize the following objective function with respect to a rotation matrix Θ:

$$\min_{\Theta \in SO(3)} \sum_{i=1}^{m} \sum_{j=1}^{n} (\triangle r_{ij} - b_i^T \Theta s_j)^2$$

where $\triangle r_{ij}$ denotes the set of differential range measurements taken at a single epoch for baseline i and satellite j, m and n respectively denote the number of baselines and satellites, $b_i \in \Re^3$ is the i-th baseline vector defined in the body frame, and $s_j \in \Re^3$ is the line of sight to the j-th satellite given in the local horizontal frame (see [6]). Θ, the optimization parameter, represents the orientation of the body frame relative to the local horizontal frame. Defining $R = (\triangle r_{ij}) \in \Re^{m\times n}$, $B = (b_1, \ldots, b_m) \in \Re^{3\times m}$, $S = (s_1, \ldots, s_n) \in \Re^{3\times n}$, the above objective function can be recast into the following form:

$$\begin{aligned} & \sum_{i=1}^{m} \sum_{j=1}^{n} (\triangle r_{ij} - b_i^T \Theta s_j)^2 \\ &= \|R - B^T \Theta S\|^2 \\ &= tr\{(R - B^T\Theta S)(R - B^T \Theta S)^T\} \\ &= tr\{\Theta N \Theta^T Q - 2\Theta W\} + tr\{RR^T\} \end{aligned}$$

where $N = SS^T$, $Q = BB^T$, and $W = SR^TB^T$, and we have made use of the general identity $tr(ABC) = tr(CAB) = tr(BCA)$. Because the last term is constant it can be neglected, so that the objective function reduces to

$$\min_{\Theta \in SO(3)} tr\{\Theta N \Theta^T Q - 2\Theta W\}$$

Observe that the objective function consists of the sum of a linear and square term in Θ, with N, Q symmetric by definition and W arbitrary.

3. STEEPEST DESCENT ALGORITHM

The steepest descent method is one of the most basic and fundamental algorithms in numerical optimization, and serves as the basis for more sophisticated optimization algorithms. Given an objective function $J(x)$ in Euclidean space, the steepest descent algorithm is given by the iteration

$$x_{k+1} = x_k - \alpha_k \nabla J(x_k)$$

where

$$\alpha_k = \arg\min_{\alpha} J(x_k - \alpha \nabla J(x_k))$$

A natural means of extending the steepest descent algorithm to functions on $SO(3)$ is to conduct the line search along the minimal geodesic in the direction of the gradient. We derive below the analytic gradient for our particular objective function, and discuss methods for performing efficient line search along the minimal geodesic.

3.1 *Computing the Gradient*

The gradient can be derived by examining the first-order term in the series expansion of the objective function. We parametrize a neighborhood of $\Theta_0 \in SO(3)$ by

$$\Theta(\Omega) = \Theta_0 \left(I + \Omega + \frac{\Omega^2}{2!} + \cdots \right)$$

with $\Omega \in so(3)$. Then to first order in Ω we have

$$\begin{aligned} & tr\{\Theta_0(I+\Omega)N(I-\Omega)\Theta_0^T Q - 2\Theta_0(I+\Omega)W\} \\ = \; & tr\{\Theta_0 N \Theta_0^T Q - 2\Theta_0 W\} + tr\{\Theta_0 \Omega N \Theta_0^T Q \\ & - \Theta_0 N \Omega \Theta_0^T Q - 2\Theta_0 \Omega W\} + o(\Omega) \end{aligned}$$

Ignoring higher-order terms, and taking advantage of the fact that N and Q are symmetric, the term linear in Ω can be simplified to

$$\langle (N\Theta_0^T Q \Theta_0 - W\Theta_0) - (N\Theta_0^T Q\Theta_0 - W\Theta_0)^T, \Omega \rangle$$

where we have also used the easily established fact that

$$tr(A\Omega) = \frac{1}{2} tr\{(A - A^T)\Omega\}$$

for any $A \in \Re^{n\times n}$ and $\Omega \in so(n)$ (i.e., $\Omega + \Omega^T = 0$). The gradient of the objective function at Θ is therefore represented by

$$\langle (N\Theta^T Q\Theta - W\Theta) - (N\Theta^T Q \Theta - W\Theta)^T, \cdot \rangle$$

and the minimal geodesic passing through Θ_0 in the direction of the gradient is given by

$$\Theta(t) = \Theta_0 e^{\Omega t},$$

where

$$\Omega = (N\Theta_0^T Q\Theta_0 - W\Theta_0) - (N\Theta_0^T Q\Theta_0 - W\Theta_0)^T$$

The steepest descent algorithm is therefore given by the following iteration:

Steepest Descent Algorithm:

$$\begin{aligned} \Omega_k &= (N\Theta_k^T Q\Theta_k - W\Theta_k) - (N\Theta_k^T Q\Theta_k - W\Theta_k)^T \\ t_k &= \min_{t\in\Re} J(\Theta_k e^{\Omega_k t}) \\ \Theta_{k+1} &= \Theta_k e^{\Omega_k t_k} \end{aligned}$$

In the next section we discuss methods for performing the line search to find the stepsize t_k.

3.2 *Line Search*

In this section we present an exact solution to the line search minimization problem, as well as an alternative formulation of Brockett's [4] method for determining a stepsize bound that ensures a decrease in the objective function.

Exact Solution In the case of $SO(3)$, because an explicit formula exists for the exponential coordinates, one can obtain the exact solution of the line search procedure by simply determining the roots of a fourth-order polynomial. The method proceeds as follows. Recalling that $e^{\Omega t} = I + \Omega \sin t + \Omega^2(1-\cos t)$ for unit length Ω, the line search minimization procedure can be expressed as

$$\min_{t\in\Re} \phi(t) = tr\{\Theta e^{\Omega t} N e^{-\Omega t}\Theta^T Q - 2\Theta e^{\Omega t} W\}$$

Observe that $\phi(t)$ is periodic with period 2π. Substituting for $e^{\Omega t}$ and simplifying, $\phi(t)$ can be expressed as

$$\phi(t) = c_1 \sin t + c_2 \cos t + c_3 \sin 2t + c_4 \cos 2t + c_5$$

with the coefficients defined as $c_1 = a + c$, $c_2 = -b - 2e$, $c_3 = -\frac{c}{2}$, $c_4 = -\frac{d}{2} + \frac{e}{2}$, and $c_5 = b + \frac{d}{2} + \frac{3e}{2} + f$, where

$$\begin{aligned} a &= tr\{(\Omega N - N\Omega)\Theta^T Q\Theta - 2\Omega W\Theta\} \\ b &= tr\{(N\Omega^2 + \Omega^2 N)\Theta^T Q\Theta - 2\Omega^2 W\Theta\} \\ c &= tr\{(\Omega N\Omega^2 - \Omega^2 N\Omega)\Theta^T Q\Theta\} \\ d &= -tr\{\Omega N\Omega\Theta^T Q\Theta\} \\ e &= tr\{\Omega^2 N\Omega^2\Theta^T Q\Theta\} \\ f &= tr\{N\Theta^T Q\Theta - 2W\Theta\} \end{aligned}$$

Making the trigonometric half-angle substitution $x = \tan\frac{x}{2}$, so that $\sin t = \frac{2x}{1+x^2}$, $\cos t = \frac{1-x^2}{1+x^2}$, $\sin 2t = \frac{4x(1-x^2)}{(1+x^2)^2}$, and $\cos 2t = \frac{x^4-6x^2+1}{(1+x^2)^2}$, finding the roots reduces to solving the following fourth-order polynomial in x:

$$\begin{aligned} (2c_3 - c_1)x^4 - 2(c_2 - 4c_4)x^3 - 12c_3 x^2 \\ - 2(c_2 + 4c_4)x + (c_1 + 2c_3) = 0 \end{aligned}$$

The roots of a fourth-order polynomial are available in most standard mathematical handbooks. Alternatively, one can employ a numerical procedure to solve the equation that involves determining the eigenvalues of a companion matrix.

Stepsize Estimate In [4] Brockett derives an estimate for the stepsize in steepest descent minimization of the function $tr(\Theta N \Theta^T Q)$. Specifically, he derives an estimate for the t that minimizes

$$\phi(t) = tr\left\{Ad_{\Theta e^{\Omega t}}(N)Q\right\}$$

for given Θ, Ω, N, Q. In this section we derive a similar stepsize estimate for the objective function $tr(\Theta N \Theta^T Q - 2\Theta W)$. The basic idea rests on the observation that, for a scalar differentiable function $f(t)$ such that $f'(0) > 0$ and $|f''(t)| \leq c$ for all t, then $f'(t) \geq 0$ in the interval $-\frac{f'(0)}{c} \leq t \leq \frac{f'(0)}{c}$. If on the other hand we have $f'(0) < 0$ and $|f''(t)| \leq c$ for all t, then $f'(t) \leq 0$ in the interval $\frac{f'(0)}{c} \leq t \leq -\frac{f'(0)}{c}$.

Let

$$\phi(t) = J(\Theta e^{\Omega t}) = tr\left\{Ad_{\Theta e^{\Omega t}}(N)Q\right\} - 2tr(\Theta e^{\Omega t} W)$$

and using the above in combination with the Cauchy-Schwarz Inequality, choosing

$$t^* = \frac{|\phi'(0)|}{\|ad_\Omega N\| \cdot \|ad_\Omega(Ad_{\Theta^T}(Q))\| + 2\|\Omega^2 W\|}$$

assures that $\phi(t^*) < \phi(0)$. Hence, at each step of the steepest descent algorithm the objective function is guaranteed to decrease. If $\phi'(0) > 0$, then setting t^* to have the opposite sign will accomplish the same result.

4. NEWTON'S METHOD

The method of steepest descent method has linear convergence, and one of its well-known disadvantages is that it converges extremely slowly near critical points where the Hessian of the objective function is poorly conditioned. Newton's method, which uses second-order terms in the Taylor expansion, in contrast has quadratic convergence. In this section we develop a Newton-type algorithm for our GPS-based attitude determination problem.

The idea behind Newton's method is that the function to be minimized is approximated locally by a quadratic function, and this approximated function is then minimized exactly. In the case of functions in Euclidean space, the function $J(x)$ is approximated near x_k by the second-order Taylor expansion

$$\begin{aligned} J(x) = {} & J(x_k) + \nabla J(x_k)(x - x_k) \\ & + \frac{1}{2}(x - x_k)^T \nabla^2 J(x_k)(x - x_k) \end{aligned}$$

where $\nabla^2 J(x) = \frac{\partial^2 J}{\partial x^2}(x)$ is the Hessian of J at x. Solving the first-order necessary conditions and adding a stepsize minimization procedure, Newton's algorithm can be expressed as

$$x_{k+1} = x_k - \alpha_k \{\nabla^2 J(x_k)\}^{-1} \nabla J(x_k)^T$$

where the stepsize α_k is chosen to minimize J along the search direction. Observe that replacing $\nabla^2 J(x_k)$ by the identity results in the steepest descent algorithm.

To generalize this method to the symmetric form of our objective function, the first order of business is to expand the objective function to second-order about some Θ_k:

$$\begin{aligned} J(\Theta) = {} & J(\Theta_k) + tr\{2(N\Theta_k^T Q \Theta_k - W\Theta_k)\Omega\} \\ & + \frac{1}{2} tr\{2(N\Theta_k^T Q\Theta_k - W\Theta_k)\Omega^2 \\ & - 2\Theta_k^T Q \Theta_k \Omega N \Omega\} + o(\Omega^2) \end{aligned}$$

where again we use left-translated exponential coordinates $\Theta = \Theta_k(I + \Omega + \frac{1}{2}\Omega^2 + \ldots)$ for the expansion. The second-order Taylor expansion is of the form

$$J(\Theta) = J(\Theta_k) + tr(F\Omega) + \frac{1}{2} tr(A\Omega^2 + B\Omega C\Omega) + o(\Omega^2)$$

where F, A, B, C are given 3×3 matrices and $\Omega \in so(3)$. Newton's algorithm can be obtained by differentiating the above with respect to Ω and setting it equal to zero, and finding the root Ω.

Omitting the details (see [9]), we can construct the following Newton-type iterative procedure for solving the GPS-based attitude determination problem. First, in this paper the following piece of notation is used. Given a vector $\omega = (\omega_1, \omega_2, \omega_3)$ and a skew-symmetric matrix $S = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$ then

$$[\omega] = S\ , \quad \vec{S} = \omega$$

Algorithm for Newton's Method:

$$\begin{aligned} T &= N\Theta_k^T Q\Theta_k + (N\Theta_k^T Q \Theta_k)^T - W\Theta_k - \Theta_k^T W \\ M &= -2\Theta_k^T Q \Theta_k \\ S &= 2\{(N\Theta_k^T Q\Theta_k)^T - N\Theta_k^T Q\Theta_k \\ & \quad + \Theta_k^T W - W\Theta_k\} \\ H &= MN + NM - (tr\ M)N - (tr\ N)M \\ & \quad + (tr\ M\ tr\ N - tr\ MN)I + (tr\ T)I - T \\ \vec{\Omega}_k &= H^{-1}\vec{S} \\ t_k &= \min_{t \in \Re} J(\Theta_k e^{\Omega_k t}) \\ \Theta_{k+1} &= \Theta_k e^{\Omega_k t_k} \end{aligned}$$

5. NUMERICAL RESULTS

We now describe results of numerical experiments for evaluating the performance of the geometric algorithms considered in this paper. One rather obvious strategy that we have not yet discussed in this paper is to simply parametrize Θ by some suitable 3-parameter representation (e.g., Euler angles, exponential coordinates) and solve the resulting unconstrained minimization problem. For comparison purposes we consider exponential coordinates, so that the objective function is of the form

$$J(x) = tr(e^{[x]}Ne^{-[x]}Q - 2We^{[x]})$$

where $\|x\| \leq \pi$.

We apply two popular unconstrained minimization algorithms provided by Matlab, the Nelder-Mead type simplex search method and the BFGS quasi-Newton method (invoked by the commands fmins() and fminu(), respectively). Because of the symbolic complexity of the objective function it is infeasible to determine analytic expressions for the gradient and Hessian in terms of local coordinates; these quantities are instead determined numerically by the Matlab routines. Also, like general user-specified Matlab programs, both fmins() and fminu() are written as m-files using Matlab's programming language. For uniform comparison the geometric algorithms of this paper are therefore also implemented as m-files.

For all the numerical experiments the elements of N, Q, and W, as well as the initial point Θ_0, are generated using the random function provided by Matlab; this generates a value between 0 and 1 of uniform probability (the effects of scaling the values of N, Q, and W are discussed below). The stopping criterion we use is $\|\Theta_{k+1} - \Theta_k\| < 10^{-4}$. Figure 1 shows the average computation times for the various algorithms, taken over 100 trials for each algorithm. In all cases the simulations are performed on a Pentium II PC with a clock speed of 266 MHz.

As evident from Figure 1, the geometric Newton's method using analytic gradients and Hessians displays the fastest average convergence, followed by the BFGS quasi-Newton method as invoked by fminu(). Unlike the traditional Newton's method for functions in Euclidean space, quasi-Newton methods employ an approximate Hessian that is much faster to compute than the actual Hessian. The algorithm with the slowest average convergence is the steepest descent method with exact stepsize. If we instead use the stepsize estimate for the line search procedure, the average convergence time improves dramatically by nearly a factor of 3.

To investigate the effects of programming language and compiler on algorithm performance, Figure 2 shows the average convergence times for the steepest descent and Newton's method, in which the algorithms are now implemented in C++. The order of performance of the three algorithms remains unchanged, although in this case there is now only a moderate difference in performance between using the exact and estimated stepsizes in steepest descent.

Finally, we observe that typical convergence times for the C++ implementation of Newton's method result in computational performance of approximately 60 cycles per second. We expect the computation times for these algorithms can be significantly reduced with improved hardware and software.

6. CONCLUSIONS

In this article we have presented a class of numerical optimization algorithms for solving the GPS-based attitude determination problem. By exploiting both the underlying geometric and algebraic structure of the rotation group $SO(3)$, efficient algorithms that generalize to $SO(3)$ the method of steepest descent and Newton's method were developed. Methods for analytically computing the exact stepsize, as well as for efficiently computing a stepsize estimate, were also outlined. Results of numerical experiments suggest that these geometric algorithms, in particular Newton's method, can offer performance advantages over traditional optimization algorithms intended for minimizing objective functions in Euclidean space.

Our benchmark for performance was to formulate the objective function in terms of the exponential coordinates, and apply several standard unconstrained minimization algorithms designed for Euclidean space. A preliminary investigation suggests that the exponential coordinates result in the fastest and most robust convergence properties vis-à-vis other local coordinate representations of $SO(3)$, e.g., Euler angles, Rodrigues parameters, unit quaternions (which introduces a constraint in this case). As of yet, however, a systematic study on the effects of the particular choice of local coordinates has yet to be carried out. Some recent related work in this connection is that of Oshman and Markley [14].

Also, as pointed out previously in the Introduction, the specific problem we address is admittedly an idealized mathematical one, and ignores a number of practical issues that must be resolved either a priori or simultaneously, e.g., cycle ambiguity resolution, the multipath problem, structural distortion due to flexure and thermal effects, tropospheric effects, and receiver-specific errors, all of which are topics of current study. Clearly there are many additional issues that need to be considered in developing a complete solution to the GPS-based attitude determination problem, but we believe the geometric perspective offers a fundamentally different and promising approach to attitude determination using GPS.

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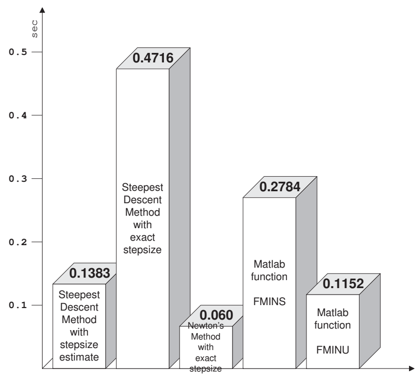

Fig. 1. Average convergence times : Matlab implementation

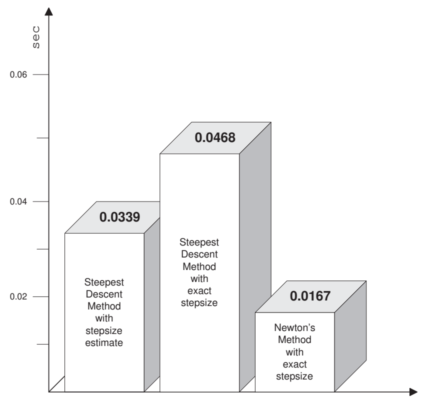

Fig. 2. Average convergence times: C++ implementation