Further Reasoning about Shared Mutable Data Structure

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Basic Ideas

- Extension of Hoare logic for imperative programs
- Low-level programming language
  - Store/heap dichotomy (both updatable)
  - Heap-manipulation commands (not expressions)
  - Heap faults
  - Nondeterministic allocation
- Independent (spatial) conjunction ($*$)
- Inductive definitions over abstract structures
- Recent extensions
  - Unrestricted address arithmetic
  - Iterated independent conjunction (arrays)
  - Assertion variables

References

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[2] Peter W. O'Hearn and David J. Pym. The logic of bunched implications. *Bulletin of Symbolic Logic*, 5(2):215–244, June 1999.

[3] John C. Reynolds. Intuitionistic reasoning about shared mutable data structure. In Jim Davies, Bill Roscoe, and Jim Woodcock, editors, *Millennial Perspectives in Computer Science*, pages 303–321, Houndsmill, Hampshire, 2000. Palgrave.

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New Commands

$$
\begin{aligned}
\langle\text{comm}\rangle ::= & \cdots \\
& \mid \langle\text{var}\rangle := \mathbf{cons}(\langle\text{exp}\rangle, \ldots, \langle\text{exp}\rangle) \\
& \mid \langle\text{var}\rangle := [\langle\text{exp}\rangle] \\
& \mid [\langle\text{exp}\rangle] := \langle\text{exp}\rangle \\
& \mid \mathbf{dispose}\, \langle\text{exp}\rangle
\end{aligned}
$$

New Assertions

$$
\begin{aligned}
\langle\text{assert}\rangle ::= & \cdots \\
& \mid \langle\text{exp}\rangle \mapsto \langle\text{exp}\rangle \\
& \mid \mathbf{emptyheap} \\
& \mid \langle\text{assert}\rangle \;*\; \langle\text{assert}\rangle \\
& \mid \langle\text{assert}\rangle \mathrel{-\!\!*} \langle\text{assert}\rangle
\end{aligned}
$$

Why **cons**, **car**, and **cdr** do not occur in expressions

$$(\mathsf{x} = \mathsf{x})/\mathsf{x} \to \mathbf{cons}(1,2) \;\text{ is }\; \mathbf{cons}(1,2) = \mathbf{cons}(1,2)$$

$$(\mathsf{x} = \mathsf{x} \;*\; \mathsf{y} = \mathsf{y})/\mathsf{x} \to \mathbf{car}(\mathsf{z}), \mathsf{y} \to \mathbf{car}(\mathsf{z}) \;\text{ is}$$

$$(\mathbf{car}(\mathsf{z}) = \mathbf{car}(\mathsf{z})) \;*\; (\mathbf{car}(\mathsf{z}) = \mathbf{car}(\mathsf{z}))$$

Stores and Heaps (Old)

$$
\begin{aligned}
\text{Values} &= \mathbf{Z} + \text{Atoms} + \text{Locations} \\
\textbf{nil} &\in \text{Atoms} \\
\text{States} &= \text{Stores} \times \text{Heaps} \\
\text{Stores} &= \langle\text{var}\rangle \to \text{Values} \\
\text{Heaps} &= \bigcup_{A \overset{\text{fin}}{\subseteq} \text{Locations}} (A \to \text{Values}^{+})
\end{aligned}
$$

Stores and Heaps (New)

$$
\begin{aligned}
\text{Atoms} &\subseteq \mathbf{Z} \\
\text{Locations} &\subseteq \mathbf{Z} \\
\text{Atoms} \cap \text{Locations} &= \{\} \\
\textbf{nil} &\in \text{Atoms} \\
\text{States} &= \text{Stores} \times \text{Heaps} \\
\text{Stores} &= \langle\text{var}\rangle \to \mathbf{Z} \\
\text{Heaps} &= \bigcup_{A \overset{\text{fin}}{\subseteq} \text{Locations}} (A \to \mathbf{Z})
\end{aligned}
$$

Semantic Equations

$$
\begin{aligned}
[\![-]\!]_{\text{exp}} &\in \langle\text{exp}\rangle \to \text{Stores} \to \mathbf{Z} \\
[\![-]\!]_{\text{boolexp}} &\in \langle\text{boolexp}\rangle \to \text{Stores} \to \mathbf{B} \\
[\![-]\!]_{\text{assert}} &\in \langle\text{assert}\rangle \to \text{States} \to \mathbf{B}
\end{aligned}
$$

Structural Operational Semantics

An execution (of a command) is a sequence

$$\gamma_0 \rightsquigarrow \gamma_1 \rightsquigarrow \cdots$$

of *configurations* γ_i related by the *transition relation* $\rightsquigarrow$.

A configuration is either:

- *nonterminal*: A command-state pair.
- *terminal*: A state or **abort**.

For example:

$$\begin{gathered}
\langle \mathsf{x} := \mathbf{cons}(1,2)\ ;\ \mathsf{y} := [\mathsf{x}]\ ;\ [\mathsf{x}+1] := 3, ([\,\mathsf{x}\colon 3 \mid \mathsf{y}\colon 4\,], [\,])\rangle \rightsquigarrow \\
\langle \mathsf{y} := [\mathsf{x}]\ ;\ [\mathsf{x}+1] := 3, ([\,\mathsf{x}\colon 37 \mid \mathsf{y}\colon 4\,], [\,37\colon 1 \mid 38\colon 2\,])\rangle \rightsquigarrow \\
\langle [\mathsf{x}+1] := 3, ([\,\mathsf{x}\colon 37 \mid \mathsf{y}\colon 1\,], [\,37\colon 1 \mid 38\colon 2\,])\rangle \rightsquigarrow \\
([\,\mathsf{x}\colon 37 \mid \mathsf{y}\colon 1\,], [\,37\colon 1 \mid 38\colon 3\,]).
\end{gathered}$$

$$\begin{gathered}
\langle \mathsf{x} := \mathbf{cons}(1,2)\ ;\ \mathsf{y} := [\mathsf{x}+2]\ ;\ [\mathsf{x}+1] := 3, ([\,\mathsf{x}\colon 3 \mid \mathsf{y}\colon 4\,], [\,])\rangle \rightsquigarrow \\
\langle \mathsf{y} := [\mathsf{x}+2]\ ;\ [\mathsf{x}+1] := 3, ([\,\mathsf{x}\colon 37 \mid \mathsf{y}\colon 4\,], [\,37\colon 1 \mid 38\colon 2\,])\rangle \rightsquigarrow \\
\mathbf{abort}.
\end{gathered}$$

Our semantics is total, but nondeterminate.

Transition Rules for the New Commands

- Construction

$$\langle v := \mathbf{cons}(e_0, \ldots, e_{n-1}), (s, h)\rangle$$
$$\rightsquigarrow ([\, s \mid v{:}\, \ell \,], [\, h \mid \ell{:}\, [\![e_0]\!]_{\mathrm{exp}} s \mid \ldots \mid \ell + n - 1{:}\, [\![e_{n-1}]\!]_{\mathrm{exp}} s \,])$$
$$\text{where } \ell, \ldots, \ell + n - 1 \in \mathrm{Locations} - \mathrm{dom}\, h$$

- Lookup

$$\langle v := [e], (s, h)\rangle \rightsquigarrow ([\, s \mid v{:}\, h([\![e]\!]_{\mathrm{exp}} s) \,], h)$$
$$\text{when } [\![e]\!]_{\mathrm{exp}} s \in \mathrm{dom}\, h$$

$$\langle v := [e], (s, h)\rangle \rightsquigarrow \mathbf{abort} \qquad \text{when } [\![e]\!]_{\mathrm{exp}} s \notin \mathrm{dom}\, h$$

- Mutation

$$\langle [e] := e', (s, h)\rangle \rightsquigarrow (s, [\, h \mid [\![e]\!]_{\mathrm{exp}} s{:}\, [\![e']\!]_{\mathrm{exp}} s \,])$$
$$\text{when } [\![e]\!]_{\mathrm{exp}} s \in \mathrm{dom}\, h$$

$$\langle [e] := e', (s, h)\rangle \rightsquigarrow \mathbf{abort} \qquad \text{when } [\![e]\!]_{\mathrm{exp}} s \notin \mathrm{dom}\, h$$

- Disposal

$$\langle \mathbf{dispose}\ e, (s, h)\rangle \rightsquigarrow (s, h \rceil (\mathrm{dom}\, h - \{[\![e]\!]_{\mathrm{exp}} s\}))$$
$$\text{when } [\![e]\!]_{\mathrm{exp}} s \in \mathrm{dom}\, h$$

$$\langle \mathbf{dispose}\ e, (s, h)\rangle \rightsquigarrow \mathbf{abort} \qquad \text{when } [\![e]\!]_{\mathrm{exp}} s \notin \mathrm{dom}\, h$$

Semantics of the New Assertions

$$[\![-]\!]_{\text{assert}} \in \langle \text{assert} \rangle \to (\text{Stores} \times \text{Heaps}) \to \mathbf{B}.$$

$[\![e \mapsto e']\!]_{\text{assert}}(s, h)$ iff

$$\operatorname{dom} h = \{[\![e]\!]_{\exp} s\} \text{ and } h([\![e]\!]_{\exp} s) = [\![e']\!]_{\exp} s,$$

$[\![\mathbf{emptyheap}]\!]_{\text{assert}}(s, h)$ iff

$$\operatorname{dom} h = \{\},$$

$[\![p_0 \ * \ p_1]\!]_{\text{assert}}(s, h)$ iff

$$\exists h_0, h_1.\ h_0 \perp h_1 \text{ and } h_0 \cdot h_1 = h \text{ iff}$$
$$\text{and } [\![p_0]\!]_{\text{assert}}(s, h_0) \text{ and } [\![p_1]\!]_{\text{assert}}(s, h_1),$$

$[\![p_0 \mathrel{-\!\!*} p_1]\!]_{\text{assert}}(s, h)$ iff

$$\forall h'.\ (h' \perp h \text{ and } [\![p_0]\!]_{\text{assert}}(s, h'))$$
$$\text{implies } [\![p_1]\!]_{\text{assert}}(s, h \cdot h'),$$

where

$$h_0 \perp h_1 \ \text{ iff } \ \operatorname{dom} h_0 \cap \operatorname{dom} h_1 = \{\}$$

$$\operatorname{dom}(h_0 \cdot h_1) \ = \ \operatorname{dom} h_0 \cup \operatorname{dom} h_1$$

$$(h_0 \cdot h_1)\ell \ = \ \begin{cases} h_0 \ell & \text{when } \ell \in \operatorname{dom} h_0 \\ h_1 \ell & \text{when } \ell \in \operatorname{dom} h_1. \end{cases}$$

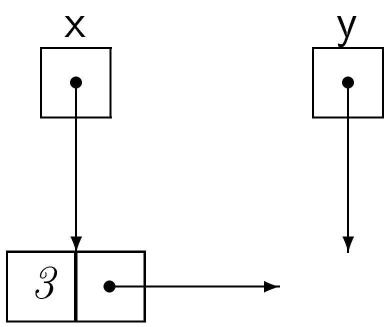
Some Abbreviations

$$
\begin{aligned}
e \mapsto - &\stackrel{\text{def}}{=} \exists x'.\ e \mapsto x' \quad \text{where } x' \text{ not free in } e \\
e \hookrightarrow e' &\stackrel{\text{def}}{=} e \mapsto e' \ast \mathbf{true} \\
e \mapsto e_0, \ldots, e_{n-1} &\stackrel{\text{def}}{=} e \mapsto e_0 \ast \cdots \ast e + n - 1 \mapsto e_{n-1} \\
e \hookrightarrow e_0, \ldots, e_{n-1} &\stackrel{\text{def}}{=} e \hookrightarrow e_0 \ast \cdots \ast e + n - 1 \hookrightarrow e_{n-1} \\
&\quad \text{iff } e \mapsto e_0, \ldots, e_{n-1} \ast \mathbf{true}
\end{aligned}
$$

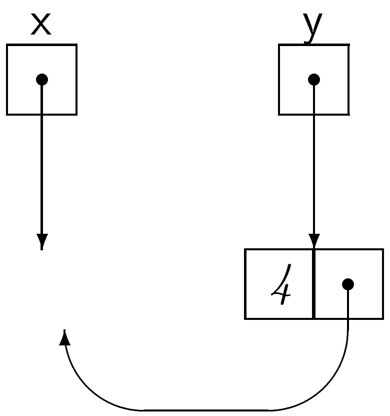
An Example of $*$

$$[\![\mathsf{x} \mapsto 3, \mathsf{y} \ * \ \mathsf{y} \mapsto 4, \mathsf{x}]\!]_{\text{assert}}(s, h)$$
$$\text{iff } \exists h_0, h_1.\ h_0 \perp h_1 \text{ and } h_0 \cdot h_1 = h$$
$$\text{and } [\![\mathsf{x} \mapsto 3, \mathsf{y}]\!]_{\text{assert}}(s, h_0) \text{ and } [\![\mathsf{y} \mapsto 4, \mathsf{x}]\!]_{\text{assert}}(s, h_1)$$

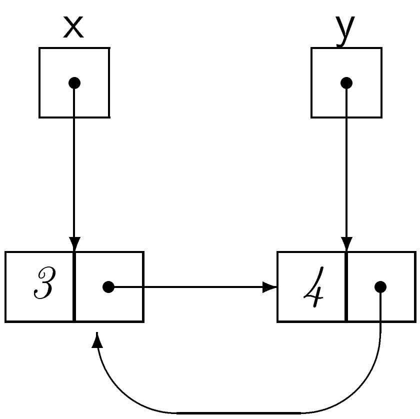
$\mathsf{x} \mapsto 3, \mathsf{y}$



$\mathsf{y} \mapsto 4, \mathsf{x}$



$\mathsf{x} \mapsto 3, \mathsf{y} \ * \ \mathsf{y} \mapsto 4, \mathsf{x}$

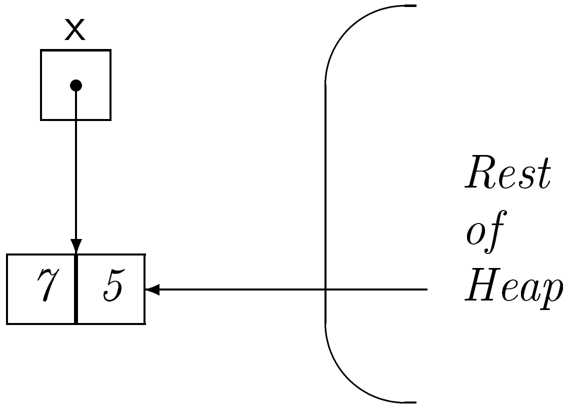


An Example of $-\!\!*$

$$\begin{aligned}&[\![\mathsf{x} \mapsto 7,5 -\!\!* P]\!]_{\mathrm{assert}}(s,h)\\ &\qquad \text{iff } \forall h'.\ (h' \perp h \text{ and } [\![\mathsf{x} \mapsto 7,5]\!]_{\mathrm{assert}}(s,h'))\\ &\qquad\qquad\qquad\qquad \text{implies } [\![P]\!]_{\mathrm{assert}}(s,h\cdot h')\end{aligned}$$

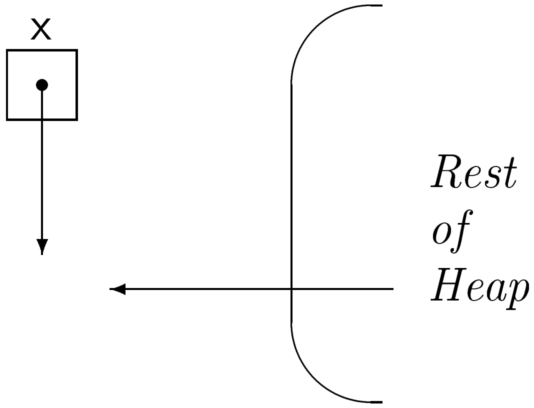
If:

P



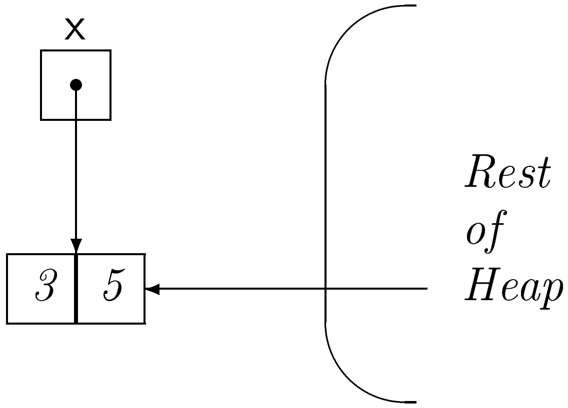
then:

$\mathsf{x} \mapsto 7,5 -\!\!* P$



and:

$\mathsf{x} \mapsto 3,5 \ * \ (\mathsf{x} \mapsto 7,5 -\!\!* P)$



Rules and Axiom Schemata for $*$ and $-\!\!*$

$$p_1 * p_2 \Leftrightarrow p_2 * p_1$$
$$(p_1 * p_2) * p_3 \Leftrightarrow p_1 * (p_2 * p_3)$$
$$p * \textbf{emptyheap} \Leftrightarrow p$$
$$(p_1 \vee p_2) * q \Leftrightarrow (p_1 * q) \vee (p_2 * q)$$
$$(p_1 \wedge p_2) * q \Rightarrow (p_1 * q) \wedge (p_2 * q)$$
$$(\exists x.\ p_1) * p_2 \Leftrightarrow \exists x.\ (p_1 * p_2) \quad \text{when } x \text{ not free in } p_2$$
$$(\forall x.\ p_1) * p_2 \Rightarrow \forall x.\ (p_1 * p_2) \quad \text{when } x \text{ not free in } p_2$$

$$\frac{p_1 \Rightarrow p_2 \qquad q_1 \Rightarrow q_2}{p_1 * q_1 \Rightarrow p_2 * q_2}$$

$$\frac{p_1 * p_2 \Rightarrow p_3}{p_1 \Rightarrow (p_2 -\!\!* p_3)} \qquad\qquad \frac{p_1 \Rightarrow (p_2 -\!\!* p_3)}{p_1 * p_2 \Rightarrow p_3}$$

Axiom Schemata for $\mapsto$

$$x \mapsto e \wedge x' \mapsto e' \Leftrightarrow x \mapsto e \wedge x = x' \wedge e = e'$$
$$x \mapsto e * x' \mapsto e' * \textbf{true} \Rightarrow x \neq x'$$
$$\textbf{emptyheap} \Leftrightarrow \forall x.\ \neg(x \mapsto - * \textbf{true}).$$

Pure Assertions

An assertion is *pure* if it does not contain $\mapsto$ or **emptyheap**.

A pure assertion does not depend upon the heap.

Axiom Schemata for Purity

$$
\begin{aligned}
p_1 \wedge p_2 \Rightarrow p_1 \ast p_2 &\quad \text{when } p_1 \text{ or } p_2 \text{ is pure} \\
p_1 \ast p_2 \Rightarrow p_1 \wedge p_2 &\quad \text{when } p_1 \text{ and } p_2 \text{ are pure} \\
(p \wedge q) \ast r \Leftrightarrow (p \ast r) \wedge q &\quad \text{when } q \text{ is pure} \\
(p_1 -\!\!\ast\ p_2) \Rightarrow (p_1 \Rightarrow p_2) &\quad \text{when } p_1 \text{ is pure} \\
(p_1 \Rightarrow p_2) \Rightarrow (p_1 -\!\!\ast\ p_2) &\quad \text{when } p_1 \text{ and } p_2 \text{ are pure}
\end{aligned}
$$

In addition, the model validates all of the laws of classical logic for $\neg$, $\Rightarrow$, $\wedge$ and $\vee$. It uses the boolean algebra structure of the powerset of Heaps, which amounts to pointwise interpretations for each classical connective.

Two Unsound Axiom Schemata

$$p \not\Rightarrow p * p \qquad \text{(Contraction)}$$

e.g. $p : \mathsf{x} \mapsto 1$

$$p * q \not\Rightarrow p \qquad \text{(Weakening)}$$

e.g. $p : \mathsf{x} \mapsto 1$

$q : \mathsf{y} \mapsto 2$

Specifications

$$[\![-]\!]_{\text{spec}} \in \langle\text{spec}\rangle \to \mathbf{B}.$$

Partial correctness:

$$\begin{aligned}
&[\![\{p\}\ c\ \{q\}]\!]_{\text{spec}} \text{ iff } \forall\sigma \in \text{States}.\ [\![p]\!]_{\text{assert}}\sigma \text{ implies}\\
&\qquad \neg\,(c, \sigma \leadsto^* \mathbf{abort})\\
&\qquad \text{and } (\forall\sigma'.\ c, \sigma \leadsto^* \sigma' \text{ implies } [\![q]\!]_{\text{assert}}\sigma')
\end{aligned}$$

Total correctness:

$$\begin{aligned}
&[\![[\ p\ ]\ c\ [\ q\ ]]\!]_{\text{spec}} \text{ iff } \forall\sigma \in \text{States}.\ [\![p]\!]_{\text{assert}}\sigma \text{ implies}\\
&\qquad \neg\,(c, \sigma \leadsto^* \mathbf{abort})\\
&\qquad \text{and } \neg\,(c, \sigma \uparrow)\\
&\qquad \text{and } (\forall\sigma'.\ c, \sigma \leadsto^* \sigma' \text{ implies } [\![q]\!]_{\text{assert}}\sigma').
\end{aligned}$$

- Specifications are implicitly quantified over both stores and heaps — and over all executions.
- An execution giving a memory fault falsifies a specification.

Examples:

$$\{\mathbf{emptyheap}\}\ \mathsf{x} := \mathbf{cons}(1, 2)\ \{\mathsf{x} \mapsto 1, 2\}$$

$$\{\mathsf{x} \mapsto 1, 2\}\ \mathsf{y} := [\mathsf{x}]\ ;\ [\mathsf{x} + 1] := 3\ \{\mathsf{x} \mapsto 1, 3 \wedge \mathsf{y} = 1\}$$

Violating Record Boundaries

Suppose P holds for a state (s, h), and C' does not assign to x. Then

$$[\,P\,]\ C\ ;\mathsf{x} := \mathbf{cons}(1,2)\ ;C'\ ;[\mathsf{x}+2] := 3\ ;C''\ [\,Q\,]$$

cannot hold. For suppose the contrary. Then

$$\langle C, (s,h)\rangle \rightsquigarrow^* (s', h').$$

Let ℓ be such that $\ell, \ell+1, \ell+2 \notin \mathrm{dom}\, h'$. Then

$$\langle \mathsf{x} := \mathbf{cons}(1,2), (s', h')\rangle \rightsquigarrow ([\,s' \mid \mathsf{x}\colon \ell\,], [\,h' \mid \ell\colon 1 \mid \ell+1\colon 2\,]),$$

and

$$\langle C', ([\,s' \mid \mathsf{x}\colon \ell\,], [\,h' \mid \ell\colon 1 \mid \ell+1\colon 2\,])\rangle \rightsquigarrow^* (s'', h''),$$

where $s''\mathsf{x} = \ell$ and $\ell + 2 \notin \mathrm{dom}\, h''$. Then

$$\langle [\mathsf{x}+2] := 3, (s'', h'')\rangle \rightsquigarrow \mathbf{abort},$$

and

$$\langle [\mathsf{x}+2] := 3\ ; C'', (s'', h'')\rangle \rightsquigarrow \mathbf{abort},$$

contradicting the assumption.

An Inference Rule for Mutation

$$\{(e \mapsto -) \ast p\}$$
$$[e] := e'$$
$$\{(e \mapsto e') \ast p\}$$

An Inference Rule for Construction

$$\{p\}$$
$$v := \mathbf{cons}(e_0, \ldots, e_{n-1})$$
$$\{(v \mapsto e_0, \ldots, e_{n-1}) \ast p\}$$

where v is not free in e_0, $\ldots$, e_{n-1}, or p.

A Stronger Rule

$$\{p\}$$
$$v := \mathbf{cons}(e_0, \ldots, e_{n-1})$$
$$\{\exists v'.\ (v \mapsto e'_0, \ldots, e'_{n-1}) \ast p'\}$$

where

- v' is distinct from v and not free in e_0, $\ldots$, e_{n-1}, or p,
- p' is the substitution $p/v \to v'$,
- each e'_i is the substitution $e_i/v \to v'$.

An Inference Rule for Lookup

$$\begin{array}{l} \{\exists v'.\ e \hookrightarrow v' \wedge p'\} \\ v := [e] \\ \{p\} \end{array}$$

where v' does not occur free in p or e, and p' is $p/v \to v'$.

Disposal

$$\begin{array}{l} \{(e \mapsto -) \ast p\} \\ \textbf{dispose}\ e \\ \{p\} \end{array}$$

Frame Rule

$$\frac{\{p\}\ c\ \{q\}}{\{p \ast r\}\ c\ \{q \ast r\}}$$

when no variable occurs both free in r and assigned by c.

Free Variable Substitution

$$\frac{\{p\}\ c\ \{q\}}{(\{p\}\ c\ \{q\})/v_1 \to e_1,\ \ldots\ ,\ v_n \to e_n},$$

where $v_1, \ldots, v_n$ are the variables occurring free in p, c, or q, and, if v_i is assigned by c, then e_i is a variable that does not occur free in any other e_j.

For example, in

$$\{\mathsf{x} = \mathsf{y}\}\ \mathsf{x} := \mathsf{x} + \mathsf{y}\ \{\mathsf{x} = 2 \times \mathsf{y}\},$$

one can substitute $\mathsf{x}, \mathsf{y}\ \to\ \mathsf{z}, 2 \times \mathsf{w} - 1$ to infer

$$\{\mathsf{z} = 2 \times \mathsf{w} - 1\}\ \mathsf{z} := \mathsf{z} + 2 \times \mathsf{w} - 1\ \{\mathsf{z} = 2 \times (2 \times \mathsf{w} - 1)\}.$$

But one cannot substitute $\mathsf{x}, \mathsf{y}\ \to\ \mathsf{z}, 2 \times \mathsf{z} - 1$ to infer

$$\{\mathsf{z} = 2 \times \mathsf{z} - 1\}\ \mathsf{z} := \mathsf{z} + 2 \times \mathsf{z} - 1\ \{\mathsf{z} = 2 \times (2 \times \mathsf{z} - 1)\}.$$

Auxiliary Variable Elimination

$$\frac{\{p\}\ c\ \{q\}}{\{\exists v.\ p\}\ c\ \{\exists v.\ q\}},$$

where v is not free in c.

For all of the new inference rules, there are similar rules for total correctness.

Simple Procedures

A procedure definition has the form

$$h(x_1, \cdots, x_m; y_1, \cdots, y_n) = c,$$

where $y_1, \cdots, y_n$ are the free variables assigned by c. and $x_1, \cdots, x_m$ are the other free variables of c.

If one can prove the specification

$$\{p\}\ c\ \{q\},$$

then one can assume the specification

$$\{p\}\ h(x_1, \cdots, x_m; y_1, \cdots, y_n)\ \{q\}$$

in proving a specification of a command that contains calls of h. (The free variables assigned by the call are $y_1, \cdots, y_n$.)

Recursion

For partial correctness, if c contains recursive calls, one can also use the assumption

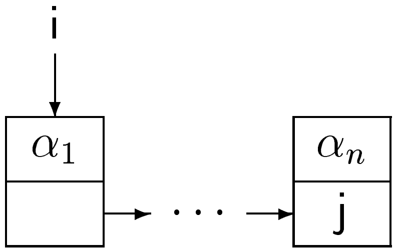
$$\{p\}\ h(x_1, \cdots, x_m; y_1, \cdots, y_n)\ \{q\}$$

in proving

$$\{p\}\ c\ \{q\}.$$

Inductive Definitions: Singly-Linked Lists

$\mathsf{list}\ \alpha\ (\mathsf{i}, \mathsf{j})$:

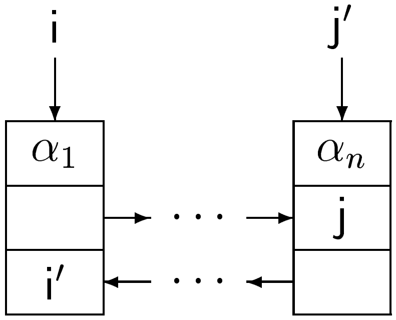


$$\mathsf{list}\ \epsilon\ (\mathsf{i}, \mathsf{j}) \stackrel{\mathrm{def}}{=} \mathbf{emptyheap} \land \mathsf{i} = \mathsf{j}$$

$$\mathsf{list}\ \mathsf{a}{\cdot}\alpha\ (\mathsf{i}, \mathsf{k}) \stackrel{\mathrm{def}}{=} \exists \mathsf{j}.\ \mathsf{i} \mapsto \mathsf{a}, \mathsf{j}\ *\ \mathsf{list}\ \alpha\ (\mathsf{j}, \mathsf{k})$$

Doubly-Linked Lists

$\mathsf{dlist}\ \alpha\ (\mathsf{i}, \mathsf{i}', \mathsf{j}, \mathsf{j}')$:



$$\mathsf{dlist}\ \epsilon\ (\mathsf{i}, \mathsf{i}', \mathsf{j}, \mathsf{j}') \stackrel{\mathrm{def}}{=} \mathbf{emptyheap} \land \mathsf{i} = \mathsf{j} \land \mathsf{i}' = \mathsf{j}'$$

$$\mathsf{dlist}\ \mathsf{a}{\cdot}\alpha\ (\mathsf{i}, \mathsf{i}', \mathsf{k}, \mathsf{k}') \stackrel{\mathrm{def}}{=} \exists \mathsf{j}.\ \mathsf{i} \mapsto \mathsf{a}, \mathsf{j}, \mathsf{i}'\ *\ \mathsf{dlist}\ \alpha\ (\mathsf{j}, \mathsf{i}, \mathsf{k}, \mathsf{k}').$$

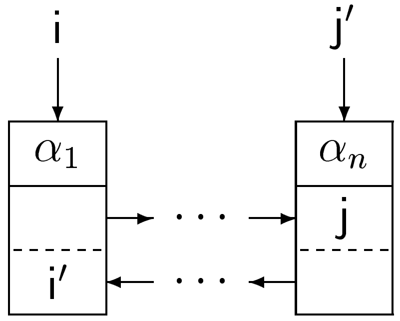
Xor-Linked Lists

$$\mathsf{xlist}\ \epsilon\ (\mathsf{i}, \mathsf{i}', \mathsf{j}, \mathsf{j}') \stackrel{\mathrm{def}}{=} \mathbf{emptyheap} \land \mathsf{i} = \mathsf{j} \land \mathsf{i}' = \mathsf{j}'$$

$$\mathsf{xlist}\ \mathsf{a}{\cdot}\alpha\ (\mathsf{i}, \mathsf{i}', \mathsf{k}, \mathsf{k}') \stackrel{\mathrm{def}}{=} \exists \mathsf{j}.\ \mathsf{i} \mapsto \mathsf{a}, (\mathsf{j} \oplus \mathsf{i}')\ *\ \mathsf{xlist}\ \alpha\ (\mathsf{j}, \mathsf{i}, \mathsf{k}, \mathsf{k}').$$

Xor-Linked Lists

$\mathsf{xlist}\,\alpha\,(\mathsf{i},\mathsf{i}',\mathsf{j},\mathsf{j}')$:



$$\mathsf{xlist}\,\epsilon\,(\mathsf{i},\mathsf{i}',\mathsf{j},\mathsf{j}') \stackrel{\mathrm{def}}{=} \mathbf{emptyheap} \wedge \mathsf{i} = \mathsf{j} \wedge \mathsf{i}' = \mathsf{j}'$$

$$\mathsf{xlist}\,\mathsf{a}{\cdot}\alpha\,(\mathsf{i},\mathsf{i}',\mathsf{k},\mathsf{k}') \stackrel{\mathrm{def}}{=} \exists \mathsf{j}.\ \mathsf{i} \mapsto \mathsf{a}, (\mathsf{j} \oplus \mathsf{i}') \ * \ \mathsf{xlist}\,\alpha\,(\mathsf{j},\mathsf{i},\mathsf{k},\mathsf{k}').$$

Properties

$$\mathsf{xlist}\,\mathsf{a}\,(\mathsf{i},\mathsf{i}',\mathsf{j},\mathsf{j}') \Leftrightarrow \mathsf{i} \mapsto \mathsf{a}, (\mathsf{j} \oplus \mathsf{i}') \wedge \mathsf{i} = \mathsf{j}'$$

$$\mathsf{xlist}\,\alpha{\cdot}\beta\,(\mathsf{i},\mathsf{i}',\mathsf{k},\mathsf{k}') \Leftrightarrow \exists \mathsf{j},\mathsf{j}'.\ \mathsf{xlist}\,\alpha\,(\mathsf{i},\mathsf{i}',\mathsf{j},\mathsf{j}') \ * \ \mathsf{xlist}\,\beta\,(\mathsf{j},\mathsf{j}',\mathsf{k},\mathsf{k}')$$

$$\mathsf{xlist}\,\alpha{\cdot}\mathsf{b}\,(\mathsf{i},\mathsf{i}',\mathsf{k},\mathsf{k}') \Leftrightarrow \exists \mathsf{j}'.\ \mathsf{xlist}\,\alpha\,(\mathsf{i},\mathsf{i}',\mathsf{k}',\mathsf{j}') \ * \ \mathsf{k}' \mapsto \mathsf{b}, (\mathsf{k} \oplus \mathsf{j}').$$

Emptyness Conditions

$$\mathsf{xlist}\,\alpha\,(\mathsf{i},\mathsf{i}',\mathsf{j},\mathsf{j}') \Rightarrow \begin{pmatrix} & (\mathsf{i} = \mathbf{nil} \Rightarrow (\alpha = \epsilon \wedge \mathsf{j} = \mathbf{nil} \wedge \mathsf{i}' = \mathsf{j}')) \\ \wedge & (\mathsf{j}' = \mathbf{nil} \Rightarrow (\alpha = \epsilon \wedge \mathsf{i}' = \mathbf{nil} \wedge \mathsf{i} = \mathsf{j})) \\ \wedge & (\mathsf{i} \neq \mathsf{j} \Rightarrow \alpha \neq \epsilon) \\ \wedge & (\mathsf{i}' \neq \mathsf{j}' \Rightarrow \alpha \neq \epsilon) \end{pmatrix}$$

A Set-pointer Procedure for Xor-linked Lists

Since

$$
\begin{array}{l}
\{\mathbf{xlist}\ \alpha\,(\mathsf{i},\mathbf{nil},\mathsf{j},\mathsf{j}')\}\\
\mathbf{if}\ \mathsf{j}' = \mathbf{nil}\ \mathbf{then}\\
\quad \{\alpha = \epsilon \wedge \mathbf{emptyheap} \wedge \mathsf{j}' = \mathbf{nil}\}\\
\quad \mathsf{i} := \mathsf{k}\\
\quad \{\alpha = \epsilon \wedge \mathbf{emptyheap} \wedge \mathsf{j}' = \mathbf{nil} \wedge \mathsf{i} = \mathsf{k}\}\\
\mathbf{else}\\
\quad \{\exists \alpha', \mathsf{b}, \mathsf{k}'.\ \ \alpha = \alpha'{\cdot}\mathsf{b} \wedge (\mathbf{xlist}\ \alpha'\,(\mathsf{i},\mathbf{nil},\mathsf{j}',\mathsf{k}')\ *\ \mathsf{j}' \mapsto \mathsf{b}, (\mathsf{j}{\oplus}\mathsf{k}'))\}\\
\quad \mathbf{newvar}\ \mathsf{x}\ \mathbf{in}\ (\mathsf{x} := [\mathsf{j}'+1]\ ;\ [\mathsf{j}'+1] := \mathsf{x} \oplus \mathsf{j} \oplus \mathsf{k})\\
\quad \{\exists \alpha', \mathsf{b}, \mathsf{k}'.\ \ \alpha = \alpha'{\cdot}\mathsf{b} \wedge (\mathbf{xlist}\ \alpha'\,(\mathsf{i},\mathbf{nil},\mathsf{j}',\mathsf{k}')\ *\ \mathsf{j}' \mapsto \mathsf{b}, (\mathsf{k}{\oplus}\mathsf{k}'))\}\\
\{\mathbf{xlist}\ \alpha\,(\mathsf{i},\mathbf{nil},\mathsf{k},\mathsf{j}')\},
\end{array}
$$

the procedure

$$
\begin{array}{l}
\mathsf{xsetrpt}(\mathsf{j},\mathsf{j}',\mathsf{k};\mathsf{i}) =\\
\quad \mathbf{if}\ \mathsf{j}' = \mathbf{nil}\ \mathbf{then}\ \mathsf{i} := \mathsf{k}\ \mathbf{else}\\
\qquad \mathbf{newvar}\ \mathsf{x}\ \mathbf{in}\ (\mathsf{x} := [\mathsf{j}'+1]\ ;\ [\mathsf{j}'+1] := \mathsf{x} \oplus \mathsf{j} \oplus \mathsf{k})
\end{array}
$$

satisfies

$$
\begin{array}{l}
\{\mathbf{xlist}\ \alpha\,(\mathsf{i},\mathbf{nil},\mathsf{j},\mathsf{j}')\}\\
\mathsf{xsetrpt}(\mathsf{j},\mathsf{j}',\mathsf{k};\mathsf{i})\\
\{\mathbf{xlist}\ \alpha\,(\mathsf{i},\mathbf{nil},\mathsf{k},\mathsf{j}')\}.
\end{array}
$$

Upon Reflection

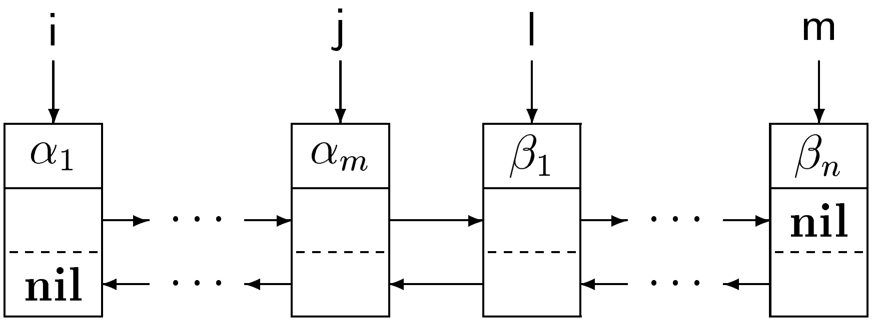
The procedure

$$\begin{array}{l} \mathsf{xsetlpt}(\mathsf{i}, \mathsf{i}', \mathsf{k}; \mathsf{j}') = \\ \quad \textbf{if } \mathsf{i} = \textbf{nil then } \mathsf{j}' := \mathsf{k} \textbf{ else} \\ \quad\quad \textbf{newvar } \mathsf{x} \textbf{ in } (\mathsf{x} := [\mathsf{i}+1]\,;\,[\mathsf{i}+1] := \mathsf{x} \oplus \mathsf{i}' \oplus \mathsf{k}) \end{array}$$

satisfies

$$\begin{array}{l} \{\textbf{xlist}\,\alpha\,(\mathsf{i}, \mathsf{i}', \textbf{nil}, \mathsf{j}')\} \\ \mathsf{xsetlpt}(\mathsf{i}, \mathsf{i}', \mathsf{k}; \mathsf{j}') \\ \{\textbf{xlist}\,\alpha\,(\mathsf{i}, \mathsf{k}, \textbf{nil}, \mathsf{j}')\}. \end{array}$$

Inserting an Element into a Xor-Linked List



$\{\textbf{xlist}\,\alpha\,(\mathsf{i},\textbf{nil},\mathsf{l},\mathsf{j})\ *\ \textbf{xlist}\,\beta\,(\mathsf{l},\mathsf{j},\textbf{nil},\mathsf{m})\}$

$\mathsf{k} := \textbf{cons}(\mathsf{a},\mathsf{l}\oplus\mathsf{j})\ ;$

$\{\textbf{xlist}\,\alpha\,(\mathsf{i},\textbf{nil},\mathsf{l},\mathsf{j})\ *\ \mathsf{k}\mapsto\mathsf{a},(\mathsf{l}\oplus\mathsf{j})\ *\ \textbf{xlist}\,\beta\,(\mathsf{l},\mathsf{j},\textbf{nil},\mathsf{m})\}$

$\mathsf{xsetrpt}(\mathsf{l},\mathsf{j},\mathsf{k};\mathsf{i})\ ;$

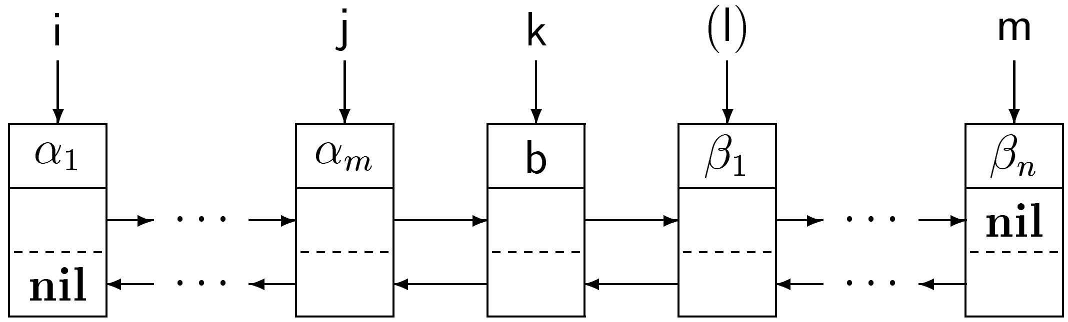
$\{\textbf{xlist}\,\alpha\,(\mathsf{i},\textbf{nil},\mathsf{k},\mathsf{j})\ *\ \mathsf{k}\mapsto\mathsf{a},(\mathsf{l}\oplus\mathsf{j})\ *\ \textbf{xlist}\,\beta\,(\mathsf{l},\mathsf{j},\textbf{nil},\mathsf{m})\}$

$\mathsf{xsetlpt}(\mathsf{l},\mathsf{j},\mathsf{k};\mathsf{m})$

$\{\textbf{xlist}\,\alpha\,(\mathsf{i},\textbf{nil},\mathsf{k},\mathsf{j})\ *\ \mathsf{k}\mapsto\mathsf{a},(\mathsf{l}\oplus\mathsf{j})\ *\ \textbf{xlist}\,\beta\,(\mathsf{l},\mathsf{k},\textbf{nil},\mathsf{m})\}$

$\{\textbf{xlist}\,\alpha{\cdot}\mathsf{a}{\cdot}\beta\,(\mathsf{i},\textbf{nil},\textbf{nil},\mathsf{m})\}$

Deleting an Element from a Xor-Linked List



$\{\exists \mathsf{l}.\ \ \mathsf{xlist}\,\alpha\,(\mathsf{i},\mathbf{nil},\mathsf{k},\mathsf{j})\ *\ \mathsf{k} \mapsto \mathsf{b}, (\mathsf{l} \oplus \mathsf{j})\ *\ \mathsf{xlist}\,\beta\,(\mathsf{l},\mathsf{k},\mathbf{nil},\mathsf{m})\}$

$\mathsf{l} := [\mathsf{k}+1] \oplus \mathsf{j}\ ;$

$\{\mathsf{xlist}\,\alpha\,(\mathsf{i},\mathbf{nil},\mathsf{k},\mathsf{j})\ *\ \mathsf{k} \mapsto \mathsf{b}, (\mathsf{l} \oplus \mathsf{j})\ *\ \mathsf{xlist}\,\beta\,(\mathsf{l},\mathsf{k},\mathbf{nil},\mathsf{m})\}$

$\mathbf{dispose}\ \mathsf{k}\ ;\ \mathbf{dispose}\ \mathsf{k}+1\ ;$

$\{\mathsf{xlist}\,\alpha\,(\mathsf{i},\mathbf{nil},\mathsf{k},\mathsf{j})\ *\ \mathsf{xlist}\,\beta\,(\mathsf{l},\mathsf{k},\mathbf{nil},\mathsf{m})\}$

$\mathbf{xsetrpt}(\mathsf{k},\mathsf{j},\mathsf{l};\mathsf{i})\ ;$

$\{\mathsf{xlist}\,\alpha\,(\mathsf{i},\mathbf{nil},\mathsf{l},\mathsf{j})\ *\ \mathsf{xlist}\,\beta\,(\mathsf{l},\mathsf{k},\mathbf{nil},\mathsf{m})\}$

$\mathbf{xsetlpt}(\mathsf{l},\mathsf{k},\mathsf{j};\mathsf{m})$

$\{\mathsf{xlist}\,\alpha\,(\mathsf{i},\mathbf{nil},\mathsf{l},\mathsf{j})\ *\ \mathsf{xlist}\,\beta\,(\mathsf{l},\mathsf{j},\mathbf{nil},\mathsf{m})\}$

$\{\mathsf{xlist}\,\alpha{\cdot}\beta\,(\mathsf{i},\mathbf{nil},\mathbf{nil},\mathsf{m})\}$

Iterated Independent Conjunction

$$\langle\text{assert}\rangle ::= \cdots \mid \bigodot_{\langle\text{var}\rangle=\langle\text{exp}\rangle}^{\langle\text{exp}\rangle} \langle\text{assert}\rangle$$

$$\begin{aligned}
&[\![\textstyle\bigodot_{x=e}^{e'} p]\!]_{\text{assert}}(s,h) = \\
&\qquad \textbf{let } m = [\![e]\!]_{\text{exp}} s,\ n = [\![e']\!]_{\text{exp}} s,\ I = \{\, i \mid m \leq i \leq n \,\} \textbf{ in} \\
&\qquad\qquad \exists H \in I \rightarrow \text{Heaps}. \\
&\qquad\qquad\qquad \forall i, j \in I.\ i \neq j \text{ implies } Hi \perp Hj \\
&\qquad\qquad\qquad \text{and } h = \bigcup\{\, Hi \mid i \in I \,\} \\
&\qquad\qquad\qquad \text{and } \forall i \in I.\ [\![p]\!]_{\text{assert}}([\, s \mid v{:}\, i \,], Hi).
\end{aligned}$$

Axiom Schemata

$$m > n \Rightarrow \Big(\textstyle\bigodot_{i=m}^{n} p(i) \Leftrightarrow \textbf{emptyheap}\Big)$$

$$m = n \Rightarrow \Big(\textstyle\bigodot_{i=m}^{n} p(i) \Leftrightarrow p(m)\Big)$$

$$k \leq m \leq n+1 \Rightarrow \Big(\textstyle\bigodot_{i=k}^{n} p(i) \Leftrightarrow \Big(\textstyle\bigodot_{i=k}^{m-1} p(i) \ * \ \textstyle\bigodot_{i=m}^{n} p(i)\Big)\Big)$$

$$\textstyle\bigodot_{i=m}^{n} p(i) \Leftrightarrow \textstyle\bigodot_{i=m-k}^{n-k} p(i+k)$$

$$m \leq n \Rightarrow \Big(\Big(\textstyle\bigodot_{i=m}^{n} p(i)\Big) \ * \ q \Leftrightarrow \textstyle\bigodot_{i=m}^{n}(p(i) \ * \ q)\Big)$$

when q is pure

$$m \leq j \leq n \Rightarrow \Big(\Big(\textstyle\bigodot_{i=m}^{n} p(i)\Big) \Rightarrow (p(j) \ * \ \textbf{true})\Big)$$

Array Allocation

$$\langle\text{comm}\rangle ::= \cdots \mid \langle\text{var}\rangle := \textbf{allocate}\ \langle\text{exp}\rangle$$

$$\langle v := \textbf{allocate}\ e, (s, h)\rangle \rightsquigarrow ([\, s \mid v{:}\, \ell\,], h \cdot h')$$
$$\text{where } h \perp h' \text{ and } \operatorname{dom} h' = \{\, i \mid \ell \le i \le \ell + [\![e]\!]_{\text{exp}} s \,\}.$$

An Inference Rule

$$\{p\}$$
$$v := \textbf{allocate}\ e$$
$$\{p \;*\; \textstyle\bigodot_{i=v}^{v+e-1} i \mapsto -\},$$

where v does not occur free in p or e.

Specifying Subset Lists

Let $\alpha, \beta \in \mathbf{Z}^*$ and $\sigma \in (\mathbf{Z}^*)^*$ in

$\{\mathsf{list}\ \alpha\ (\mathsf{i}, \mathbf{nil})\}$
"Set j to list of lists of subsets of i"
$\{\exists \sigma, \beta.\ \ \mathsf{ss}(\alpha^\dagger, \sigma) \wedge (\mathsf{list}\ \beta\ (\mathsf{j}, \mathbf{nil})\ * \ (Q(\sigma, \beta) \wedge R(\beta)))\}$,

where

$$\begin{aligned}
\#\mathsf{ext_a}\sigma &\stackrel{\text{def}}{=} \#\sigma \\
(\mathsf{ext_a}\sigma)_i &\stackrel{\text{def}}{=} \mathsf{a}{\cdot}\sigma_i
\end{aligned}$$

$$\begin{aligned}
\mathsf{ss}(\epsilon, \sigma) &\stackrel{\text{def}}{=} \sigma = [\epsilon] \\
\mathsf{ss}(\mathsf{a}{\cdot}\alpha, \sigma) &\stackrel{\text{def}}{=} \exists \sigma'.\ \ \mathsf{ss}(\alpha, \sigma') \wedge \sigma = (\mathsf{ext_a}\sigma')^\dagger{\cdot}\sigma'
\end{aligned}$$

$$Q(\sigma, \beta) \stackrel{\text{def}}{=} \#\beta = \#\sigma \wedge \forall_{\mathsf{i}=1}^{\#\beta}(\mathsf{list}\ \sigma_\mathsf{i}\ (\beta_\mathsf{i}, \mathbf{nil})\ *\ \mathbf{true})$$

$$\begin{aligned}
R(\beta) \stackrel{\text{def}}{=}\ & (\beta_{\#\beta} = \mathbf{nil} \wedge \mathbf{emptyheap})\ * \\
& \bigodot_{\mathsf{i}=1}^{\#\beta-1}(\exists \mathsf{a}, \mathsf{k}.\ \ \mathsf{i} < \mathsf{k} \leq \#\beta \wedge \beta_\mathsf{i} \mapsto \mathsf{a}, \beta_\mathsf{k}).
\end{aligned}$$

Trees (Binary)

$\tau \in \text{Trees}$ iff

$$\tau \in \text{Atoms}$$

$$\text{or } \tau = (\tau_1, \tau_2) \text{ where } \tau_1, \tau_2 \in \text{Trees}.$$

Tree Representation (without sharing)

For $\tau \in \text{Trees}$, we define the assertion

$$\mathsf{tree}\,\tau\,(i)$$

by structural induction:

$$\mathsf{tree}\,a\,(i) \text{ iff } \mathbf{emptyheap} \wedge i = a$$

$$\begin{aligned}&\mathsf{tree}\,(\tau_1, \tau_2)\,(i) \text{ iff}\\ &\quad \exists i_1, i_2.\ \ i \mapsto i_1, i_2 \ * \ \mathsf{tree}\,\tau_1\,(i_1) \ * \ \mathsf{tree}\,\tau_2\,(i_2).\end{aligned}$$

Tree Representation (with sharing)

For $\tau \in \text{Trees}$, we define

$$\mathsf{stree}\,\tau\,(i)$$

by:

$$\mathsf{stree}\,a\,(i) \text{ iff } i = a$$

$$\begin{aligned}&\mathsf{stree}\,(\tau_1, \tau_2)\,(i) \text{ iff}\\ &\quad \exists i_1, i_2.\ \ i \mapsto i_1, i_2 \ * \ (\mathsf{stree}\,\tau_1\,(i_1) \wedge \mathsf{stree}\,\tau_2\,(i_2)).\end{aligned}$$

A Problem

Suppose we wish to prove that a recursively defined procedure $\mathsf{copytree}$ satisfies

$$\{\mathsf{stree}\,\tau(\mathsf{i})\}$$
$$\mathsf{copytree}(\mathsf{i};\mathsf{j})$$
$$\{\mathsf{stree}\,\tau(\mathsf{i}) \ * \ \mathsf{tree}\,\tau(\mathsf{j})\}$$

Then, we must use this specification as a hypothesis in proving that the recursive calls in the procedure body satisfy:

$$\{\mathsf{stree}\,\tau_1(\mathsf{i}_1) \wedge \mathsf{stree}\,\tau_2(\mathsf{i}_2)\}$$
$$\mathsf{copytree}(\mathsf{i}_1;\mathsf{j}_1)$$
$$\{(\mathsf{stree}\,\tau_1(\mathsf{i}_1) \wedge \mathsf{stree}\,\tau_2(\mathsf{i}_2)) \ * \ \mathsf{tree}\,\tau_1(\mathsf{j}_1)\}$$
$$\mathsf{copytree}(\mathsf{i}_2;\mathsf{j}_2)$$
$$\{(\mathsf{stree}\,\tau_1(\mathsf{i}_1) \wedge \mathsf{stree}\,\tau_2(\mathsf{i}_2)) \ * \ \mathsf{tree}\,\tau_1(\mathsf{j}_1) \ * \ \mathsf{tree}\,\tau_2(\mathsf{j}_2)\}$$

But the hypothesis is not strong enough to imply this.

Using Assertion Variables

The procedure

$$
\begin{array}{l}
\mathsf{copytree}(\mathsf{i};\mathsf{j}) = \\
\quad \textbf{if } \mathsf{isatom}(\mathsf{i}) \textbf{ then } \mathsf{j} := \mathsf{i} \textbf{ else} \\
\quad\quad \textbf{newvar } \mathsf{i}_1, \mathsf{i}_2, \mathsf{j}_1, \mathsf{j}_2 \textbf{ in} \\
\quad\quad\quad \big(\mathsf{i}_1 := [\mathsf{i}] \,;\, \mathsf{i}_2 := [\mathsf{i}+1] \,; \\
\quad\quad\quad \textbf{copytree}(\mathsf{i}_1;\mathsf{j}_1) \,;\, \textbf{copytree}(\mathsf{i}_2;\mathsf{j}_2) \,;\, \mathsf{j} := \textbf{cons}(\mathsf{j}_1,\mathsf{j}_2)\big)
\end{array}
$$

satisfies

$$
\begin{array}{l}
\{\mathsf{p} \wedge \mathsf{stree}\,\tau(\mathsf{i})\} \\
\mathsf{copytree}(\mathsf{i};\mathsf{j}) \\
\{\mathsf{p} \,*\, \mathsf{tree}\,\tau(\mathsf{j})\}
\end{array}
$$

since

$$
\begin{array}{l}
\{\mathsf{p} \wedge \mathsf{stree}\,\tau(\mathsf{i})\} \\
\textbf{if } \mathsf{isatom}(\mathsf{i}) \textbf{ then} \\
\quad \{\mathsf{p} \wedge \mathsf{isatom}(\tau) \wedge \tau = \mathsf{i}\} \\
\quad \{\mathsf{p} \ast (\mathsf{isatom}(\tau) \wedge \tau = \mathsf{i} \wedge \mathbf{emptyheap})\} \\
\quad \mathsf{j} := \mathsf{i} \\
\quad \{\mathsf{p} \ast (\mathsf{isatom}(\tau) \wedge \tau = \mathsf{j} \wedge \mathbf{emptyheap})\} \\
\textbf{else} \\
\quad \{\exists \tau_1, \tau_2.\ \tau = (\tau_1, \tau_2) \wedge \mathsf{p} \wedge \mathsf{stree}\,(\tau_1, \tau_2)(\mathsf{i})\} \\
\quad \textbf{newvar } \mathsf{i}_1, \mathsf{i}_2, \mathsf{j}_1, \mathsf{j}_2 \textbf{ in} \\
\qquad \big(\mathsf{i}_1 := [\mathsf{i}]\,;\, \mathsf{i}_2 := [\mathsf{i} + 1]\,; \\
\qquad \{\exists \tau_1, \tau_2.\ \tau = (\tau_1, \tau_2) \wedge \\
\qquad\quad \mathsf{p} \wedge (\mathsf{i} \mapsto \mathsf{i}_1, \mathsf{i}_2 \ast (\mathsf{stree}\,\tau_1\,(\mathsf{i}_1) \wedge \mathsf{stree}\,\tau_2\,(\mathsf{i}_2)))\} \\
\qquad \{\exists \tau_1, \tau_2.\ \tau = (\tau_1, \tau_2) \wedge \mathsf{p} \wedge \mathsf{stree}\,\tau_1\,(\mathsf{i}_1) \wedge \mathsf{stree}\,\tau_2\,(\mathsf{i}_2)\} \\
\qquad \mathbf{copytree}(\mathsf{i}_1; \mathsf{j}_1)\,; \\
\qquad \{\exists \tau_1, \tau_2.\ (\tau = (\tau_1, \tau_2) \wedge \mathsf{p} \wedge \mathsf{stree}\,\tau_2\,(\mathsf{i}_2)) \ast \mathsf{tree}\,\tau_1\,(\mathsf{j}_1)\} \\
\qquad \mathbf{copytree}(\mathsf{i}_2; \mathsf{j}_2)\,; \\
\qquad \{\exists \tau_1, \tau_2.\ (\tau = (\tau_1, \tau_2) \wedge \mathsf{p}) \ast \mathsf{tree}\,\tau_1\,(\mathsf{j}_1) \ast \mathsf{tree}\,\tau_2\,(\mathsf{j}_2)\} \\
\qquad \mathsf{j} := \mathbf{cons}(\mathsf{j}_1, \mathsf{j}_2) \\
\qquad \{\exists \tau_1, \tau_2.\ (\tau = (\tau_1, \tau_2) \wedge \mathsf{p}) \ast \\
\qquad\quad \mathsf{j} \mapsto \mathsf{j}_1, \mathsf{j}_2 \ast \mathsf{tree}\,\tau_1\,(\mathsf{j}_1) \ast \mathsf{tree}\,\tau_2\,(\mathsf{j}_2)\} \\
\qquad \{\exists \tau_1, \tau_2.\ (\tau = (\tau_1, \tau_2) \wedge \mathsf{p}) \ast \mathsf{tree}\,(\tau_1, \tau_2)\,(\mathsf{j})\}\big) \\
\{(\mathsf{p} \ast \mathsf{tree}\,\tau(\mathsf{j})\}
\end{array}
$$

Future Directions

- Higher-order logic and syntactic control of interference
- Concurrency (and passivity)
- Regions
- Axiomatizing directed graphs
- Garbage collection
- Connections with type systems - relative pointers
- Embedding code pointers in data structures