Automata on Infinite Objects

Seminar Presentation by
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In this seminar we will look at various automata on infinite words (or \( \omega \)-words). In particular:

- Büchi Automata (BA)

- Second Order Theory of One Successor (S1S)

- Muller automata (MA)

- Büchi’s & McNaughtons Theorem

- DBA, NBA, DMA, NMA, \( \omega \)-words
\[ \sum = \{a, b, c\} \]

- Note for a string to be accepted, it should visit s1 infinitely often

- So the above BA accepts all strings \( \alpha \) in which \( a \) occurs infinitely often
Büchi automata

Formally, it is a tuple $\mathcal{A} = (Q, q_0, \delta, F)$, where

- $Q$ is the finite state set
- $q_0 \in Q$ is the starting state
- $\delta$ is transition relation, $\delta \subseteq Q \times A \times Q$
- $F \subseteq Q$ is the set of final states.
- $\omega$-word is accepted if some final state is visited infinitely many times.
Closure Properties of Büchi Recognizable languages

- If $V \subseteq A^*$ is regular then $V^w$ is Büchi recognizable.

- Closed under prefixing with normal finite automata.

- Closed under Union, Intersection & Complementation
Intersection example

BA $A_1$

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<tr>
<td>s1</td>
<td>s2</td>
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BA $A_2$

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<td>s1</td>
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BA $A$ s.t. $L(A) = L(A_1) \cap L(A_2)$

- Final states $= \{(s1, s3, 1), (s1, s4, 1)\}$
Complementation of BA

- BA are closed under complementation

- Complementation of DBA can be done in $2 \times n$ states.

- Complementation of NBA may require $c^{n^2}$ states. (Proof involves expressing $L$ and $L^c$ as finite union of equivalence classes)

- DBA’s not closed under complementation.

- $L(DBA) \subseteq L(NBA)$
Complementation of DBA example

DBA $\mathcal{A}$ which is to be complemented

$\mathcal{A}_1$ & $\mathcal{A}_2$, two modified copies of $\mathcal{A}$

$\mathcal{A}^c$ obtained after combining $\mathcal{A}_1$ & $\mathcal{A}_2$ as above
Second Order Theory of One Successor (S1S)

Used to specify properties of $w$-sequences.

- A variable can be one of $x, y, \ldots$. Used for positions
- A set variable can be $X, Y, \ldots$
- We have a function symbol $+1$ (successor)
- Terms, Atomic formulas, $S1A_A$ formulas
\textbf{SIS formulas examples}

- \( \sum = \{a, b, c\} \)

- Infinitely many \( a's \)

\[
\ldots a \ldots a \ldots a \ldots a \ldots a \ldots
\]

\( \varphi_1 : \ \forall x \exists y(y > x \land y \in Q_a) \)

- Two \( a's \) followed by \( b \)

\[
\ldots aab \ldots aab \ldots abc \ldots \ldots aab \ldots bbb \ldots
\]

\( \varphi_2 : \ \forall x \forall y(x \in Q_a \land y \in Q_a \land y = x + 1 \rightarrow \\
\exists z (z = y + 1 \land z \in Q_b)) \)
Büchi’s Theorem

**Theorem 1** An \( w \)-language is definable in S1S iff it is regular

- BA to S1S formula
- S1S formula to BA

**Theorem 2** Truth of sentences of SIS is decidable

Decidability of S1S was one of the motivations for Büchi.
\[ \mathcal{F} = \{\{s_1\}, \{s_1, s_2, s_3\}\} \]

- For \( \alpha = acbacb \ldots \) the Infinity set is \( \{s_1, s_2, s_3\} \). So \( \alpha \) is Muller acceptable.

- For \( \alpha = ababab \ldots \) the Infinity set is \( \{s_1, s_2\} \). So \( \alpha \) is not Muller acceptable.
Theorem 3  An \( \omega \)-language is regular (i.e, Büchi recognizable) iff it is Muller recognizable.

Consequences
Conclusions

- Expresssive equivalence of BA, S1S, MA
- Acceptance criterion
- Union, Intersection
- Complementation of DBA Vs. NBA
- Complementation of MA
Applications

- Decidability of S1S formulas
- Behaviour of certain digital circuits
- Real Time Systems
- Programs, Specifications, Concurrent programs
Example 3

BA for $X_1 \subseteq X_2$

BA for $\varphi : \exists X_1 \exists X_2 \neg (X_1 \subseteq X_2)$
Determinism & McNaughton’s Theorem

- We can show DBA’s are not closed under complement.

- And hence DBA’s are strictly weaker then NBA in general

- By refining the acceptance condition we can have deterministic automata which are as powerful as NBA’s.

- One such deterministic version is

- Muller Automaton