Ordered Adjoint Logic

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Abstract. Ordered logics and type systems have been used in a variety of applications including computational linguistics, memory allocation, stream processing, logical frameworks, parametricity, and enforcing security protocols. In most formulations, ordered types are also linear, requiring each resource to be used exactly once. Prior work by Kanovich et al. has investigated calculi that relax this constraint through subexponentials within a linear ordered logic. We generalize their work by using adjoint modalities to combine logics with varying fine-grained structural properties, including weakening, left contraction, right contraction, left mobility, and right mobility. We show that the resulting sequent calculus admits cut elimination.
We further provide a natural deduction formulation in which structural rules are implicit, and show that proof checking for this system is decidable. This makes it a suitable foundation for an expressive adjoint programming language or logical framework.

1 Introduction

Gentzen's original sequent calculus [8] had explicit structural rules for exchange, weakening, and contraction. If we freely allow exchange, and control the uses of weakening and contraction via an exponential modality, we obtain either classical or intuitionistic linear logic [9]. Prior to that, Lambek [15] had considered a sequent calculus that disallows exchange, driven by applications in linguistics. A combination of the ideas is natural, and has been considered for the classical [33, 2, 30, 17] and intuitionistic [22, 23] cases. The latter has found applications in logical frameworks [24, 21, 20, 31, 19], stream processing [7, 6, 4], and semantic characterization of programs via parametricity [1].

In this paper we focus on the intuitionistic variant, where we use the term *ordered* for logics lacking exchange. In many applications, we need to combine several forms of substructural reasoning, which has been controlled with a modality for mobility $¡A$ (effectively making A linear), and the usual $!A$ that also allowing weakening and contraction [23]. This has been generalized by Kanovich et al. [13, 14] using *subexponentials* [5, 18] to give an open-ended calculus parameterized by a preorder of modes that may or may not satisfy the structural properties of mobility, weakening, and contraction.

A drawback of subexponentials is that they are interpreted relative to a base mode in which all the logical inferences take place. For ordered logics, this base mode must prohibit all structural rules. As a result, applications often require switching into and out of the base mode. Benton's LNL [3] offers a solution by allowing linear and nonlinear reasoning to be connected via two adjoint modalities. This means we can reason natively in ordinary or purely linear intuitionistic logic and employ modalities only where it is necessary to include one in the other. This has been generalized to *adjoint logic* [28, 26, 16, 27, 25], essentially by applying Benton's decomposition of the exponential of linear logic to subexponentials. A related decomposition has been applied to graded modal logic [11, 32], although not with explicit control of exchange as far as we are aware.

In this paper we introduce ordered adjoint logic, which features a preorder of modes, each of which may or may not satisfy mobility, weakening, or contraction. Instead of a particular base mode (as with subexponentials), we use the *shift modalities* from adjoint logic to switch between reasoning at different modes. We further decompose mobility and contraction to be one-sided, a possibility anticipated but not explored in the subexponential sequent calculus of Kanovich et al. [14].

We study ordered adjoint logic from two perspectives. We start with a sequent calculus and show that it satisfies cut and identity elimination, establishing its bona fides as a well-behaved logical system. We then introduce a system of natural deduction for ordered adjoint logic that generalizes the adjoint types of Jang et al. [12], also controlling mobility. The result of this latter step is a natural deduction system that can easily be expanded to a type system for a functional programming language in which one can postulate and freely combine rather fine-grained information about the structural properties of code. This extension to programming languages would follow a bidirectional type checking discipline with the set of terms equivalent to those in a non-ordered setting. We show that the proof-checking problem for this system is decidable even when all applications of structural rules remain implicit.

The same structures recur for a substructural language based on a Kripke semantics with explicit worlds called QRTT [10], which has been applied to represent and enforce various security properties via typing. Consequently, our result applies to ensure decidability of *substructural subsumption*, which is at the core of the type-checking algorithm for QRTT.

We present an ordered adjoint sequent calculus in section 2. We conclude this section with proofs of cut and identity elimination. In section 3, we introduce two natural deduction calculi: one with explicit structural rules and one where they remain implicit. We prove decidability of proof checking for the latter. A more detailed presentation can be found in the related version [29].

2 Ordered Adjoint Sequent Calculus

In his seminal work on LNL, Benton [3] connects purely linear intuitionistic logic with ordinary intuitionistic logic with two adjoint modalities: F that includes unrestricted formulas in the linear ones, and G that includes the linear formulas in the unrestricted ones. The beauty of this approach is that, under the Curry-Howard isomorphism, one can natively program in either linear or nonlinear discipline, and switch between them only as needed. Moreover, we can recover the $!A$ as $F\,G\,A$. This was later generalized in the form of *adjoint logic* [28, 26, 25], which is parameterized by a preorder $k \geq m$ of modes, each of which satisfies a set of structural properties $\sigma(m) \subseteq \{\mathsf{W}, \mathsf{C}\}$. Here, W stands for *weakening*, C for *contraction*, while *exchange* is always assumed. Structural properties must be monotonic, that is, $k \geq m$ implies $\sigma(k) \supseteq \sigma(m)$. Propositions are indexed by modes A_m, and F and G are generalized to *shifts* $(\downarrow^k_m A_k)_m$ and $(\uparrow^m_l A_l)_m$. LNL is a special case with two modes $\mathsf{U} > \mathsf{L}$, $\sigma(\mathsf{U}) = \{\mathsf{W}, \mathsf{C}\}$ and $\sigma(\mathsf{L}) = \{\,\}$. Furthermore, $G\,A$ becomes $\uparrow^{\mathsf{U}}_{\mathsf{L}} A_{\mathsf{L}}$ and $F\,X$ becomes $\downarrow^{\mathsf{U}}_{\mathsf{L}} X_{\mathsf{U}}$. We make the presupposition that in a sequent $\Gamma \vdash A_m$, each antecedent B_k in Γ must satisfy $k \geq m$. In particular, $B_{\mathsf{L}} \vdash A_{\mathsf{U}}$, would be semantically unsound: via a cut we could derive both weakening and contraction for B_{L}. We abbreviate this *declaration of independence* as $\Gamma \geq m$. Remarkably, all the logical rules are parametric in their modes. Their specific structural properties only come into play when we have to decide whether a structural rule can be applied to an antecedent.

In this section we further generalize adjoint logic to account for order among antecedents. Everything explained above about adjoint logic still applies.

2.1 Structural Rules in an Ordered Setting

In Gentzen's presentation of sequent calculus, reordering of antecedents is governed by the exchange rule, which allows two adjacent elements to be swapped.

$$\frac{\Omega_L\ B\ A\ \Omega_R \vdash C}{\Omega_L\ A\ B\ \Omega_R \vdash C}\ \mathsf{Exchg}$$

Exchange acts on two propositions. In the setting of adjoint logic, however, structural rules apply only when the mode of a proposition admits the corresponding structural property. This raises the question, when should this swap be allowed?

An elegant solution is offered by Kanovich et al. [14]: rather than trying to enforce some reasonable condition on this rule they use *mobility*. Instead of swapping two propositions, mobility rules move a single proposition arbitrarily far to the left or right within the antecedents. This formulation aligns naturally with the unary presentation of weakening and contraction. Each mobility rule is permitted precisely when the mode of the proposition admits the corresponding directional mobility. New here is that we treat left and right mobility separately.

Since it is important to track specific occurrences of hypotheses in sequents, we label each antecedent with a variable. When new antecedents are introduced in the premises of rules, we choose names not already present in the sequent. Here, and subsequently, we use Ω to stand for the collection of antecedents instead of Γ in order to emphasize that their order is significant.

$$\frac{\Omega_L\ \Omega_M\ (x{:}A_m)\ \Omega_R \vdash C_r \quad \mathsf{M}^{\leftarrow} \in \sigma(m)}{\Omega_L\ (x{:}A_m)\ \Omega_M\ \Omega_R \vdash C_r}\ \mathsf{M}^{\leftarrow}$$

$$\frac{\Omega_L\ (x{:}A_m)\ \Omega_M\ \Omega_R \vdash C_r \quad \mathsf{M}^{\rightarrow} \in \sigma(m)}{\Omega_L\ \Omega_M\ (x{:}A_m)\ \Omega_R \vdash C_r}\ \mathsf{M}^{\rightarrow}$$

If all antecedents have the same structural properties, left mobility implies right mobility and vice versa. However, immobile antecedents can block exchange, so the two structural properties become independent. This has applications, for example, in modeling protocols and security properties (see section 4).

As observed by Kanovich et al., if we restrict contraction to two adjacent antecedents the resulting calculus does not satisfy cut elimination. We therefore admit non-local contraction in both directions. As with mobility, we maintain two separate structural properties.

$$\frac{\Omega_L\,(x{:}A_m)\,\Omega_M\,(x{:}A_m)\,\Omega_R \vdash C_r \quad \mathsf{C}^{\leftarrow} \in \sigma(m)}{\Omega_L\,(x{:}A_m)\,\Omega_M\,\Omega_R \vdash C_r}\ \mathsf{C}^{\leftarrow}$$

$$\frac{\Omega_L\,(x{:}A_m)\,\Omega_M\,(x{:}A_m)\,\Omega_R \vdash C_r \quad \mathsf{C}^{\rightarrow} \in \sigma(m)}{\Omega_L\,\Omega_M\,(x{:}A_m)\,\Omega_R \vdash C_r}\ \mathsf{C}^{\rightarrow}$$

Tracking contraction in this manner with duplicate variables will be important in the proof of cut admissibility and in section 3. While duplicate variables are allowed, all occurrences of a variable must label the same proposition, and variables introduced in the premises of rules of must be fresh. Weakening remains a single rule:

$$\frac{\Omega_L \Omega_R \vdash C_r \quad \mathsf{W} \in \sigma(m)}{\Omega_L\,(x{:}A_m)\,\Omega_R \vdash C_r}\ \mathsf{W}$$

2.2 Cut and Identity

The rule of identity is unremarkable. The rule of cut requires a condition on modes so that our presupposition is preserved when reading the rule from the conclusion to the premises. We will shortly have occasion to generalize cut in order to prove its admissibility in the cut-free sequent calculus.

$$\frac{}{x{:}A_m \vdash A_m}\ \mathsf{id} \qquad \frac{\Omega \vdash A_m \quad \Omega_L\,(x{:}A_m)\,\Omega_R \vdash C_r \quad (\Omega \geq m \geq r)}{\Omega_L\,\Omega\,\Omega_R \vdash C_r}\ \mathsf{cut}$$

2.3 Logical Rules

As compared to linear logic, implication splits into two connectives: left implication $A_m \rightarrowtail B_m$ (written $A \backslash B$ in the Lambek calculus [15]) and right implication $A_m \twoheadrightarrow B_m$ (usually written $B \,/\, A$). It is also possible to split conjunction into fuse ($A_m \bullet B_m$) and twist ($A_m \circ B_m$). The latter is usually omitted since $A_m \bullet B_m$ is isomorphic to $B_m \circ A_m$. We therefore obtain the following language of propositions, where P_m stands for atomic propositions. Note that in the first row, all propositions have rules which are invertible on the right and therefore are negative, while in the second row they have rules which are invertible on the left and therefore are positive.

$$\begin{array}{rll} A_m, B_m ::= & P_m \mid A_m \rightarrowtail B_m \mid A_m \twoheadrightarrow B_m \mid A_m \,\&\, B_m \mid {\uparrow}_l^m A_l & (m \geq l) \\ \mid & A_m \bullet B_m \mid \mathbf{1}_m \mid A_m \oplus B_m \mid {\downarrow}_m^k A_k & (k \geq m) \end{array}$$

Remarkably, as for unordered adjoint logic, the left and right rules for the connectives are entirely parametric in their mode. We just have to be more careful about maintaining order. We highlight the rules for left and right implication. The complete set of rules can be found in Figure 1.

$$\dfrac{\Omega\,(x{:}A_m) \vdash B_m}{\Omega \vdash A_m \twoheadrightarrow B_m}\,{\twoheadrightarrow}R \qquad \dfrac{\Omega_A \vdash A_m \quad \Omega_A \geq m \quad \Omega_L\,(y{:}B_m)\,\Omega_R \vdash C_r}{\Omega_L\,(f{:}A_m \twoheadrightarrow B_m)\,\Omega_A\,\Omega_R \vdash C_r}\,{\twoheadrightarrow}L$$

$$\dfrac{(x{:}A_m)\,\Omega \vdash B_m}{\Omega \vdash A_m \rightarrowtail B_m}\,{\rightarrowtail}R \qquad \dfrac{\Omega_A \vdash A_m \quad \Omega_A \geq m \quad \Omega_L\,(y{:}B_m)\,\Omega_R \vdash C_r}{\Omega_L\,\Omega_A\,(f{:}A_m \rightarrowtail B_m)\,\Omega_R \vdash C_r}\,{\rightarrowtail}L$$

The rules regarding shifts are trickier, but the conditions on modes are systematically derived from the need to preserve *independence* when reading the rules from the conclusion to the premises.

The rules for the downshift are:

$$\dfrac{\Omega_L\,(y{:}A_k)\,\Omega_R \vdash C_r}{\Omega_L\,(x{:}{\downarrow}^k_m A_k)\,\Omega_R \vdash C_r}\,{\downarrow}L \qquad \dfrac{\Omega \vdash A_k \quad (\Omega \geq k)}{\Omega \vdash {\downarrow}^k_m A_k}\,{\downarrow}R$$

In the left rule, the presuppositions $\Omega_L\,\Omega_R \geq r$ and $k \geq m \geq r$ ensure that the premise is well formed. In the right rule, $\Omega \geq m$ and $k \geq m$ do not imply $\Omega \geq k$, so this condition must be checked explicitly.

The upshift rules are:

$$\dfrac{\Omega_L\,(y{:}A_l)\,\Omega_R \vdash C_r \quad (l \geq r)}{\Omega_L\,(x{:}{\uparrow}^m_l A_l)\,\Omega_R \vdash C_r}\,{\uparrow}L \qquad \dfrac{\Omega \vdash A_l}{\Omega \vdash {\uparrow}^m_l A_l}\,{\uparrow}R$$

For the left rule, the condition $l \geq r$ must be verified explicitly, since it does not follow from $\Omega_L\,\Omega_R \geq r$, $m \geq l$ and $m \geq r$. The right rule has no additional conditions on modes. To show where unidirectional mobility makes a difference, consider the following proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{y{:}B_m \vdash B_m}\,\mathsf{id} \quad \dfrac{\dfrac{}{x'{:}A_k \vdash A_k}\,\mathsf{id} \quad (k \geq k)}{x'{:}A_k \vdash {\downarrow}^k_m A_k}\,{\downarrow}R}{(y{:}B_m)\,(x'{:}A_k) \vdash B_m \bullet {\downarrow}^k_m A_k}\,{\bullet}R \qquad \mathsf{M}^{\leftarrow} \in \sigma(k)}{(x'{:}A_k)\,(y{:}B_m) \vdash B_m \bullet {\downarrow}^k_m A_k}\,\mathsf{M}^{\leftarrow}}{(x{:}{\downarrow}^k_m A_k)\,(y{:}B_m) \vdash B_m \bullet {\downarrow}^k_m A_k}\,{\downarrow}L}{x{:}{\downarrow}^k_m A_k \vdash B_m \twoheadrightarrow B_m \bullet {\downarrow}^k_m A_k}\,{\twoheadrightarrow}R}{\cdot \vdash {\downarrow}^k_m A_k \twoheadrightarrow B_m \twoheadrightarrow B_m \bullet {\downarrow}^k_m A_k}\,{\twoheadrightarrow}R$$

This proof goes through if k admits left mobility (m need not admit any mobility), or if m admits any mobility (in which case we replace the $\mathsf{M}^{\leftarrow}$ of x' with an $\mathsf{M}^{\rightarrow}$ of y). However, there is no proof if m is immobile and k only admits right mobility.

$$\frac{}{\cdot \vdash 1_m}\ 1R \qquad \frac{\Omega_L\ \Omega_R \vdash C_r}{\Omega_L\ (x{:}1_m)\ \Omega_R \vdash z{:}C_r}\ 1L$$

$$\frac{\Omega_1 \vdash A_m \quad \Omega_2 \vdash B_m}{\Omega_1\ \Omega_2 \vdash A_m \bullet B_m}\ \bullet R \qquad \frac{\Omega_L\ (x_1{:}A_m)\ (x_2{:}B_m)\ \Omega_R \vdash C_r}{\Omega_L\ (x{:}A_m \bullet B_m)\ \Omega_R \vdash C_r}\ \bullet L$$

$$\frac{\Omega \vdash A_m}{\Omega \vdash A_m \oplus B_m}\ \oplus R_1 \qquad \frac{\Omega \vdash B_m}{\Omega \vdash A_m \oplus B_m}\ \oplus R_2$$

$$\frac{\Omega_L\ (y{:}A_m)\ \Omega_R \vdash C_r \quad \Omega_L\ (y{:}B_m)\ \Omega_R \vdash C_r}{\Omega_L\ (x{:}A_m \oplus B_m)\ \Omega_R \vdash C_r}\ \oplus L$$

$$\frac{\Omega \vdash A_k \quad (\Omega \geq k)}{\Omega \vdash {\downarrow}^k_m A_k}\ {\downarrow}R \qquad \frac{\Omega_L\ (y{:}A_k)\ \Omega_R \vdash C_r}{\Omega_L\ (x{:}{\downarrow}^k_m A_k)\ \Omega_R \vdash C_r}\ {\downarrow}L$$

$$\frac{\Omega\ (x{:}A_m) \vdash B_m}{\Omega \vdash A_m \twoheadrightarrow B_m}\ \twoheadrightarrow R \qquad \frac{\Omega_A \vdash A_m \quad \Omega_A \geq m \quad \Omega_L\ (y{:}B_m)\ \Omega_R \vdash C_r}{\Omega_L\ (f{:}A_m \twoheadrightarrow B_m)\ \Omega_A\ \Omega_R \vdash C_r}\ \twoheadrightarrow L$$

$$\frac{(x{:}A_m)\ \Omega \vdash B_m}{\Omega \vdash A_m \rightarrowtail B_m}\ \rightarrowtail R \qquad \frac{\Omega_A \vdash A_m \quad \Omega_A \geq m \quad \Omega_L\ (y{:}B_m)\ \Omega_R \vdash C_r}{\Omega_L\ \Omega_A\ (f{:}A_m \rightarrowtail B_m)\ \Omega_R \vdash C_r}\ \rightarrowtail L$$

$$\frac{\Omega \vdash A_m \quad \Omega \vdash B_m}{\Omega \vdash A_m \mathbin{\&} B_m}\ \&R$$

$$\frac{\Omega_L\ (y{:}A_m)\ \Omega_R \vdash C_r}{\Omega_L\ (x{:}A_m \mathbin{\&} B_m)\ \Omega_R \vdash C_r}\ \&L_1 \qquad \frac{\Omega_L\ (y{:}B_m)\ \Omega_R \vdash C_r}{\Omega_L\ (x{:}A_m \mathbin{\&} B_m)\ \Omega_R \vdash C_r}\ \&L_2$$

$$\frac{\Omega \vdash A_l}{\Omega \vdash {\uparrow}^m_l A_l}\ {\uparrow}R \qquad \frac{\Omega_L\ (y{:}A_l)\ \Omega_R \vdash C_r \quad (l \geq r)}{\Omega_L\ (x{:}{\uparrow}^m_l A_l)\ \Omega_R \vdash C_r}\ {\uparrow}L$$

$$\frac{}{x{:}A_m \vdash A_m}\ \mathsf{id} \qquad \frac{\Omega_A \vdash A_m \quad \Omega_L\ (x{:}A_m)\ \Omega_R \vdash C_r \quad (\Omega \geq m \geq r)}{\Omega_L\ \Omega_A\ \Omega_R \vdash C_r}\ \mathsf{cut}$$

Fig. 1. Ordered Adjoint Sequent Calculus (structural rules in subsection 2.1)

2.4 Cut Elimination

Because we have explicit contraction rules, with multiple copies of the same variable appearing in a context, we use a form of multicut rather than a single cut [8, 9, 25]. The number n of occurrences of a variable replaced in the antecedents must be compatible with the structural properties of the mode m, written $|m| \sim n$. Specifically:

$$\begin{array}{l} |m| \sim 0 \text{ if } \mathsf{W} \in \sigma(m) \\ |m| \sim 1 \text{ always} \\ |m| \sim n \text{ for } n > 1 \text{ if } \mathsf{C}^{\leftarrow} \in \sigma(m) \text{ or } \mathsf{C}^{\rightarrow} \in \sigma(m) \end{array}$$

In multicut, $\Omega(x{:}A_m)^n$ denotes n occurrences of $x{:}A_m$ distributed throughout Ω, and $\Omega(\Omega_A)^n$ denotes their replacement by Ω_A. By our general assumptions, the variable x must be fresh in the premise.

$$\frac{\Omega_A \vdash A_m \quad \Omega\,(x{:}A_m)^n \vdash C_r \quad (\Omega_A \geq m \geq r) \quad |m| \sim n}{\Omega(\Omega_A)^n \vdash C_r}\ \mathsf{mcut}$$

We can now state and prove the admissibility of multicut in the cut-free calculus, followed by general cut elimination.

Theorem 1 (Admissibility of Multicut). *If $\Omega_A \vdash A_m$ and $\Omega(x{:}A_m)^n \vdash C_r$ with cut-free derivations, then $\Omega(\Omega_A)^n \vdash C_r$ has a cut-free derivation.*

Proof. We proceed by induction on the ordered triple $(A_m, \mathcal{D}, \mathcal{E})$ We show the principal cases for weakening, and one direction of contraction. We use dashed lines to indicate admissibility.

Case: $\mathcal{E}$ ends in W on one occurrence of the principal formula.

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Omega_A \vdash A_m\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}' \\ (\Omega_L\ \Omega_R)(x{:}A_m)^{n-1} \vdash C_r\end{array} \quad \begin{array}{c}(2)\\ \mathsf{W} \in \sigma(m)\end{array}}{(\Omega_L\ (x{:}A_m)\ \Omega_R)(x{:}A_m)^{n-1} \vdash C_r}\ \mathsf{W} \quad \begin{array}{c}(1)\\ \Omega_A \geq m \geq r\end{array}}{(\Omega_L\ \Omega_A\ \Omega_R)(\Omega_A)^{n-1} \vdash C_r}\ \mathsf{mcut}$$

$(\Omega_L\ \Omega_R)(\Omega_A)^{n-1} \vdash C_r$	by IH on $(A_m, \mathcal{D}, \mathcal{E}')$, with (1)
Ω_A admits weakening	by monotonicity ((2),(1))
$(\Omega_L\ \Omega_A\ \Omega_R)(\Omega_A)^{n-1} \vdash C_r$	by W on elements of Ω_A

Case: $\mathcal{E}$ ends in $\mathsf{C}^{\leftarrow}$ on (one copy of) principal formula.

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Omega_A \vdash A_m\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}' \\ (\Omega_L\ (x{:}A_m)\ \Omega_M\ (x{:}A_m)\ \Omega_R)(x{:}A_m)^{n-1} \vdash C_r\end{array} \quad (2)}{(\Omega_L\ (x{:}A_m)\ \Omega_M\ \Omega_R)(x{:}A_m)^{n-1} \vdash C_r}\ \mathsf{C}^{\leftarrow} \quad (1)}{(\Omega_L\ \Omega_A\ \Omega_M\ \Omega_R)(\Omega_A)^{n-1} \vdash C_r}\ \mathsf{mcut}$$

With

$$(1) = \Omega_A \geq m \geq r \quad (2) = \mathsf{C}^{\leftarrow} \in \sigma(m)$$

$(\Omega_L\ \Omega_A\ \Omega_M\ \Omega_A\ \Omega_R)(\Omega_A)^{n-1} \vdash C_r$	by IH on $(A_m, \mathcal{D}, \mathcal{E}')$, with (1)
Ω_A admits left contraction	by monotonicity $((2), (1))$
$(\Omega_L\ \Omega_A\ \Omega_M\ \Omega_R)(\Omega_A)^{n-1} \vdash C_r$	by $\mathsf{C}^{\leftarrow}$ on elements of Ω_A

Theorem 2 (Cut Elimination). *If $\Omega \vdash C_r$ then there is a cut-free proof of this sequent.*

Proof. By straightforward structural induction on the given derivation. For the case of cut, we apply admissibility of multicut to the result of eliminating cut from the derivations of the premises.

As a second test for the system, the identity rule should be admissible except for atoms. As usual, this is straightforward in a sequent calculus.

Theorem 3 (Admissibility of Identity). *The rule of identity is admissible in a system where it is restricted to atomic propositions.*

Proof. By induction over the structure of the proposition A_m.

3 Natural Deduction

We now move towards a presentation of a natural deduction system in which structural rules are implicit. We do so in a natural deduction context rather than staying within the sequent calculus as the final goal of this work is to have a type checker for an extension of the adjoint natural deduction system in [12] that considers order. However, for the purpose of this paper we elide proof terms and show the decidability of natural deduction proof checking in the absence of explicit structural rules instead. The final step to bidirectional type-checking is then rather straightforward and is patterned after [12].

It is perhaps somewhat unexpected that proof checking with implicit structural rules is not easily seen to be decidable, as is the case in unordered adjoint natural deduction. The culprits are the interactions of unidirectional mobility, weakening, and the nondeterminism in locating the new antecedents in the positive elimination rules. The corresponding problem in the sequent calculus is that in the cut rule, the new antecedent $x{:}A_m$ can be anywhere in the context, a property that is required for cut elimination.

We begin with a non-algorithmic system in subsection 3.1, which still keeps structural rules explicit to build a bridge to the sequent calculus. In subsection 3.2 we present a terminating algorithm to check proofs with implicit structural rules. Optimizations beyond the scope of this paper are still necessary to make this algorithm practical and are briefly discussed in the conclusion.

3.1 Explicit Structural Rules

We begin with the judgment

$$\Gamma \vdash A_m \dashv \Omega$$

where Γ is an unordered set of all hypotheses $y{:}B_k$ that are syntactically in scope while proving A_m. Ω is the context of the ones that are actually used, in order.

As usual in natural deduction, rules are divided into introduction and elimination rules. We consider $\bullet I$ as a first sample rule.

$$\frac{\Gamma \vdash A_m \dashv \Omega_A \quad \Gamma \vdash B_m \dashv \Omega_B}{\Gamma \vdash A_m \bullet B_m \dashv \Omega_A\, \Omega_B}\ \bullet I$$

Ω_A and Ω_B are the ordered lists of labeled hypothesis used in the proofs of A_m and B_m, respectively. They must be concatenated to form the hypotheses used to prove the pair.

In the presence of contraction, the same labeled hypothesis $y{:}B_k$ may be present in both Ω_A and Ω_B. In unordered adjoint (and other substructural logics), they can be directly merged, but here we require an explicit application of a contraction rule, if permissible.

$$\frac{\Gamma \vdash C_r \dashv \Omega_L\, (x{:}A_m)\, \Omega_M\, (x{:}A_m)\, \Omega_R \quad \mathsf{C}^{\rightarrow} \in \sigma(m)}{\Gamma \vdash C_r \dashv \Omega_L\, \Omega_M\, (x{:}A_m)\, \Omega_R}\ \mathsf{C}^{\rightarrow}$$

$$\frac{\Gamma \vdash C_r \dashv \Omega_L\, (x{:}A_m)\, \Omega_M\, (x{:}A_m)\, \Omega_R \quad \mathsf{C}^{\leftarrow} \in \sigma(m)}{\Gamma \vdash C_r \dashv \Omega_L\, (x{:}A_m)\, \Omega_M\, \Omega_R}\ \mathsf{C}^{\leftarrow}$$

The $\bullet E$ rule is one source of considerable nondeterminism. We write $|\Omega|$ for the set of variables declared in Ω.

$$\frac{\Gamma \vdash A_m \bullet B_m \dashv \Omega \quad (m \geq r) \quad \begin{array}{r}(x, y \notin |\Omega_L\, \Omega_R|)\\ \Gamma, x{:}A_m, y{:}B_m \vdash C_r \dashv \Omega_L\, (x{:}A_m)\, (y{:}B_m)\, \Omega_R\end{array}}{\Gamma \vdash C_r \dashv \Omega_L\, \Omega\, \Omega_R}\ \bullet E$$

One difficulty is that due to unidirectional mobility and also due to weakening, the location for x and y in the output context of the second premise may not be uniquely determined. Furthermore, x and y go out of scope, so they are not allowed to occur in Ω_L and Ω_R (nor in Ω by our general freshness assumption).

We also have rules where two output contexts must match. $\& I$ is shown:

$$\frac{\Gamma \vdash A_m \dashv \Omega \quad \Gamma \vdash B_m \dashv \Omega}{\Gamma \vdash A_m \,\&\, B_m \dashv \Omega}\ \& I$$

This might require that some weakening occur before application of this rule in the first and second premise (assuming the mode m permits it).

The structural rules for weakening and mobility on the output context are:

$$\frac{\Gamma \vdash C_r \dashv \Omega_L \Omega_R \quad x{:}A_m \in \Gamma \quad (m \geq r) \quad \mathsf{W} \in \sigma(m)}{\Gamma \vdash C_r \dashv \Omega_L (x{:}A_m) \Omega_R}\ \mathsf{W}$$

$$\frac{\Gamma \vdash C_r \dashv \Omega_L\, (x{:}A_m)\, \Omega_M\, \Omega_R \quad \mathsf{M}^{\rightarrow} \in \sigma(m)}{\Gamma \vdash C_r \dashv \Omega_L\, \Omega_M\, (x{:}A_m)\, \Omega_R}\ \mathsf{M}^{\rightarrow}$$

$$\frac{\Gamma \vdash C_r \dashv \Omega_L\, \Omega_M\, (x{:}A_m)\, \Omega_R \quad \mathsf{M}^{\leftarrow} \in \sigma(m)}{\Gamma \vdash C_r \dashv \Omega_L\, (x{:}A_m)\, \Omega_M\, \Omega_R}\ \mathsf{M}^{\leftarrow}$$

$$\frac{}{\Gamma \vdash 1_m \dashv \cdot}\ 1I \qquad \frac{\Gamma \vdash 1_m \dashv \Omega_M \quad (m \geq r) \quad \Gamma \vdash C_r \dashv \Omega_L\ \Omega_R}{\Gamma \vdash C_r \dashv \Omega_L\ \Omega_M\ \Omega_R}\ 1E$$

$$\frac{\Gamma \vdash A_m \dashv \Omega_L \quad \Gamma \vdash B_m \dashv \Omega_R}{\Gamma \vdash A_m \bullet B_m \dashv \Omega_L \Omega_R}\ \bullet I$$

$$\frac{\Gamma \vdash A_m \bullet B_m \dashv \Omega_M \quad (m \geq r) \quad \begin{array}{r}(x, y \notin \Omega_L\ \Omega_R)\\ \Gamma, x{:}A_m, y{:}B_m \vdash C_r \dashv \Omega_L\ (x{:}A_m)\ (y{:}B_m)\ \Omega_R\end{array}}{\Gamma \vdash C_r \dashv \Omega_L\ \Omega_M\ \Omega_R}\ \bullet E$$

$$\frac{\Gamma \vdash A_m \dashv \Omega}{\Gamma \vdash A_m \oplus B_m \dashv \Omega}\ \oplus I_1 \qquad \frac{\Gamma \vdash B_m \dashv \Omega}{\Gamma \vdash A_m \oplus B_m \dashv \Omega}\ \oplus I_2$$

$$\frac{\Gamma \vdash A_m \oplus B_m \dashv \Omega_M \quad (m \geq r) \quad \begin{array}{r}x \notin |\Omega_L\ \Omega_R|\\ \Gamma, x{:}A_m \vdash C_r \dashv \Omega_L\ (x{:}A_m)\ \Omega_R\\ \Gamma, x{:}B_m \vdash C_r \dashv \Omega_L\ (x{:}B_m)\ \Omega_R\end{array}}{\Gamma \vdash C_r \dashv \Omega_L\ \Omega_M\ \Omega_R}\ \oplus E$$

$$\frac{\Gamma \vdash A_k \dashv \Omega}{\Gamma \vdash {\downarrow}^k_m A_k \dashv \Omega}\ {\downarrow}I$$

$$\frac{\Gamma \vdash {\downarrow}^k_m A_k \dashv \Omega_M \quad (m \geq r) \quad \Gamma, x{:}A_k \vdash C_r \dashv \Omega_L\ (x{:}A_k)\ \Omega_R \quad x \notin |\Omega_L\ \Omega_R|}{\Gamma \vdash C_r \dashv \Omega_L\ \Omega_M\ \Omega_R}\ {\downarrow}E$$

$$\frac{\Gamma, x{:}A_m \vdash B_m \dashv \Omega\ (x{:}A_m) \quad x \notin \Omega}{\Gamma \vdash A_m \twoheadrightarrow B_m \dashv \Omega}\ {\twoheadrightarrow} I \qquad \frac{\Gamma \vdash A_m \twoheadrightarrow B_m \dashv \Omega_L \quad \Gamma \vdash A_m \dashv \Omega_R}{\Gamma \vdash B_m \dashv \Omega_L \Omega_R}\ {\twoheadrightarrow} E$$

$$\frac{\Gamma, x{:}A_m \vdash B_m \dashv (x{:}A_m)\ \Omega \quad x \notin \Omega}{\Gamma \vdash A_m \rightarrowtail B_m \dashv \Omega}\ {\rightarrowtail} I \qquad \frac{\Gamma \vdash A_m \rightarrowtail B_m \dashv \Omega_R \quad \Gamma \vdash A_m \dashv \Omega_L}{\Gamma \vdash B_m \dashv \Omega_L\ \Omega_R}\ {\rightarrowtail} E$$

$$\frac{\Gamma \vdash A_m \dashv \Omega \quad \Gamma \vdash B_m \dashv \Omega}{\Gamma \vdash A_m \,\&\, B_m \dashv \Omega}\ \&I \qquad \frac{\Gamma \vdash A_m \,\&\, B_m \dashv \Omega}{\Gamma \vdash A_m \dashv \Omega}\ \&E_1 \qquad \frac{\Gamma \vdash A_m \,\&\, B_m \dashv \Omega}{\Gamma \vdash B_m \dashv \Omega}\ \&E_2$$

$$\frac{\Gamma \vdash A_l \dashv \Omega \quad \Omega \geq m}{\Gamma \vdash {\uparrow}^m_l A_l \dashv \Omega}\ {\uparrow}I \qquad \frac{\Gamma \vdash {\uparrow}^l_k A_k \dashv \Omega}{\Gamma \vdash A_k \dashv \Omega}\ {\uparrow}E$$

$$\frac{x{:}A_m \in \Gamma}{\Gamma \vdash A_m \dashv (x{:}A_m)}\ \mathsf{hyp}$$

Fig. 2. Ordered Natural Deduction (explicit structural rules in subsection 3.1)

The remaining logical rules can be found in Figure 2.

We relate this system to the sequent calculus defined in section 2. There are more fine-grained translations (say, relating cut-free sequent derivations to normal natural deduction; see [12] for the unordered adjoint case), but this is not relevant for our purposes here.

Lemma 1 (Substitution).
1. *If* $\Gamma \vdash A_m \dashv \Omega$ *then* $\Gamma \supseteq \Omega$ *and* $\Omega \geq m$
2. *If* $\Gamma \vdash A_m \dashv \Omega$ *and* $\Gamma' \supseteq \Gamma$ *then* $\Gamma' \vdash A_m \dashv \Omega$
3. *If* $\Gamma_1 \vdash A_m \dashv \Omega_A$ *and* $\Gamma_2, x{:}A_m \vdash C_r \dashv \Omega(x{:}A_m)^n$ *then* $\Gamma_1 \cup \Gamma_2 \vdash C_r \dashv \Omega(\Omega_A)^n$ *provided* $|m| \sim n$.

Proof. All by simple rule inductions on given derivations.

Theorem 4 (Sequent Calculus to Natural Deduction).
If $\Omega \vdash A_m$ *then* $\Gamma \vdash A_m \dashv \Omega$ *for all* $\Gamma \supseteq \Omega$

Proof. By induction on the given derivation, using substitution in several cases.

Theorem 5 (Natural Deduction to Sequent Calculus).
If $\Gamma \vdash A_m \dashv \Omega$ *then* $\Omega \vdash A_m$.

Proof. By induction on the structure of the given derivation, using cut and identity in several cases.

3.2 Implicit Structural Rules

If these explicit structural rules from the last section are elided, checking the validity of the remaining proof skeleton is not immediately decidable. One issue arises from the rule of weakening

$$\frac{\Gamma \vdash C_r \dashv \Omega_L\, \Omega_R \quad (m \geq r) \quad x{:}A \in \Gamma \quad \mathsf{W} \in \sigma(m)}{\Gamma \vdash C_r \dashv \Omega_L\, (x{:}A_m)\, \Omega_R}\ \mathsf{W}$$

which could be applied arbitrarily often. To address this issue, we take advantage of the following property, observed by [14]: mobility is derivable from weakening and contraction. The following shows how to derive left mobility from left contraction and weakening:

$$\frac{\dfrac{\Gamma \vdash C_r \dashv \Omega_L\, \Omega_M\, (x{:}A_m)\, \Omega_R \quad (m \geq r) \quad (x{:}A) \in \Gamma \quad \mathsf{W} \in \sigma(m)}{\Gamma \vdash C_r \dashv \Omega_L\, (x{:}A_m)\, \Omega_M\, (x{:}A_m)\, \Omega_R}\ \mathsf{W} \quad \mathsf{C}^{\leftarrow} \in \sigma(m)}{\Gamma \vdash C_r \dashv \Omega_L\, (x{:}A_m)\, \Omega_M\, \Omega_R}\ \mathsf{C}^{\leftarrow}$$

While we could just use this derived rule, it is convenient to assume the following two closure properties:

$$\text{If } \mathsf{W} \in \sigma(m) \text{ and } \mathsf{C}^{\leftarrow} \in \sigma(m) \text{ then } \mathsf{M}^{\leftarrow} \in \sigma(m) \tag{1}$$

$$\text{If } \mathsf{W} \in \sigma(m) \text{ and } \mathsf{C}^{\rightarrow} \in \sigma(m) \text{ then } \mathsf{M}^{\rightarrow} \in \sigma(m) \tag{2}$$

Under conditions (1) and (2), we can restrict weakening to only be allowed when the variable is not yet present in the output as follows:

$$\frac{\Gamma \vdash C_r \dashv \Omega_L\, \Omega_R \quad (m \geq r) \quad (x{:}A_m) \in \Gamma \quad \mathsf{W} \in \sigma(m) \quad x \notin |\Omega_L\, \Omega_R|}{\Gamma \vdash C_r \dashv \Omega_L\, (x{:}A_m)\, \Omega_R}\ \mathsf{W}$$

To construct a system with implicit structural rules that has a decidable algorithm for proof checking, we bundle the structural rules into a new judgment $\Omega \gg_{\Gamma,r} \Omega_{\mathsf{nf}}$ where Ω_{nf} denotes a *normal form* where no variable is repeated. We write Ω nf for the judgment that holds iff Ω is normal in this sense. We carry the unordered Γ and mode r in order to control weakening, with the presupposition that $\Gamma \supseteq \Omega$ and $\Omega \geq r$ and ensure that $\Gamma \supseteq \Omega_{\mathsf{nf}}$ and $\Omega_{\mathsf{nf}} \geq r$. Algorithmically, we view the left-hand side Ω together with Γ and r as inputs and Ω_{nf} as output.

$$\frac{\Omega_L(x{:}A_m)\Omega_R \gg_{\Gamma,r} \Omega_{\mathsf{nf}} \quad \mathsf{W} \in \sigma(m) \quad m \geq r \quad x{:}A_m \in \Gamma \quad x \notin |\Omega_L \Omega_R|}{\Omega_L \Omega_R \gg_{\Gamma,r} \Omega_{\mathsf{nf}}}\ \mathsf{W}$$

$$\frac{\Omega_L(x{:}A_m)\Omega_M\Omega_R \gg_{\Gamma,r} \Omega_{\mathsf{nf}} \quad \mathsf{C}^{\leftarrow} \in \sigma(m)}{\Omega_L(x{:}A_m)\Omega_M(x{:}A_m)\Omega_R \gg_{\Gamma,r} \Omega_{\mathsf{nf}}}\ \mathsf{C}^{\leftarrow}$$

$$\frac{\Omega_L\Omega_M(x{:}A_m)\Omega_R \gg_{\Gamma,r} \Omega_{\mathsf{nf}} \quad \mathsf{C}^{\rightarrow} \in \sigma(m)}{\Omega_L(x{:}A_m)\Omega_M(x{:}A_m)\Omega_R \gg_{\Gamma,r} \Omega_{\mathsf{nf}}}\ \mathsf{C}^{\rightarrow}$$

$$\frac{\Omega_L(x{:}A_m)\Omega_M\Omega_R \gg_{\Gamma,r} \Omega_{\mathsf{nf}} \quad \mathsf{M}^{\leftarrow} \in \sigma(m)}{\Omega_L\Omega_M(x{:}A_m)\Omega_R \gg_{\Gamma,r} \Omega_{\mathsf{nf}}}\ \mathsf{M}^{\leftarrow}$$

$$\frac{\Omega_L\Omega_M(x{:}A_m)\Omega_R \gg_{\Gamma,r} \Omega_{\mathsf{nf}} \quad \mathsf{M}^{\rightarrow} \in \sigma(m)}{\Omega_L(x{:}A_m)\Omega_M\Omega_R \gg_{\Gamma,r} \Omega_{\mathsf{nf}}}\ \mathsf{M}^{\rightarrow}$$

$$\frac{(\Omega\ \mathsf{nf})}{\Omega \gg_{\Gamma,r} \Omega}\ \mathsf{id}$$

For any given Ω, Γ, and r, there may be many Ω_{nf} such that $\Omega \gg_{\Gamma,r} \Omega_{\mathsf{nf}}$. To account for this we define

$$\mathsf{NF}_{\Gamma,r}(\Omega) = \{\Omega_{\mathsf{nf}} \mid \Omega \gg_{\Gamma,r} \Omega_{\mathsf{nf}}\}$$

For example, given $\Gamma = x{:}A_m, y{:}B_k$, $\Omega = y{:}B_k$ with $m \geq r$, $k \geq r$, and $\sigma(m) = \{\mathsf{W}\}$ we obtain $\mathsf{NF}_{\Gamma,r}(\Omega) = \{(y : B_k), (x : A_m)(y : B_k), (y : B_k)(x : A_m)\}$

Since Γ and the mode r are usually immediately apparent from context, we often omit them. Fortunately, $\mathsf{NF}_{\Gamma,r}(\Omega)$ is always finite because the only rule that increases the size of the context (reading bottom-up) is weakening, and it is restricted to variables that are in Γ but not yet in the input context Ω.

We define a new judgment $\Gamma \vdash A_m \dashv\!\!\!\dashv \Xi$, where Ξ is a *set of ordered contexts* in normal form. The guiding principle is embodied in Theorem 9: If $\Gamma \vdash A_m \dashv \Omega$ then $\Gamma \vdash A_m \dashv\!\!\!\dashv \Xi$ where all normal forms of Ω are contained in Ξ.

Logical rules are defined via set construction. Because the output of the judgment $\Gamma \vdash A_m \dashv\!\!\!\dashv \Xi$ is a set of contexts, we lift the normal form operation to sets, defining

$$\mathsf{NF}_{\Gamma,r}(\Xi) = \bigcup_{\Omega \in \Xi} \{\Omega' \mid \Omega' \in \mathsf{NF}_{\Gamma,r}(\Omega)\}$$

Consider the $\bullet I$ rule (We write $\Xi_1\,\Xi_2$ for $\{\Omega_1\,\Omega_2 \mid \Omega_1 \in \Xi_1, \Omega_2 \in \Xi_2\}$):

$$\frac{\Gamma \vdash A_m \dashv\!\!\!\dashv \Xi_1 \quad \Gamma \vdash B_m \dashv\!\!\!\dashv \Xi_2}{\Gamma \vdash A_m \bullet B_m \dashv\!\!\!\dashv \mathsf{NF}_{\Gamma,m}(\Xi_1\,\Xi_2)}\ \bullet I$$

The $\twoheadrightarrow I$ rule becomes:

$$\frac{\Gamma, x{:}A_m \vdash B_m \dashv\!\!\!\dashv \Xi' \quad \Xi' \| \Xi(x{:}A_m)}{\Gamma \vdash A_m \twoheadrightarrow B_m \dashv\!\!\!\dashv \Xi}\ \twoheadrightarrow I$$

$\Xi' \| \Xi(x{:}A_m)$ filters Ξ' to only contexts that end in $x{:}A_m$. Then in the conclusion $\Xi = \{\Omega \mid \Omega\,(x{:}A_m) \in \Xi'\}$. Similarly, we write $\Xi\|_m = \{\Omega \in \Xi \mid \Omega \geq m\}$. This is used in the $\uparrow I$ rule. Implicitly, all rules fail if no output contexts are possible. The last operator is related to the case where multiple output contexts must be equivalent, that is the $\& I$ rule and the $\oplus E$ rule. We walk through $\& I$:

$$\frac{\Gamma \vdash A_m \dashv\!\!\!\dashv \Xi_1 \quad \Gamma \vdash B_m \dashv\!\!\!\dashv \Xi_2}{\Gamma \vdash A_m \mathbin{\&} B_m \dashv\!\!\!\dashv \Xi_1 \cap \Xi_2}\ \& I$$

Since Ξ_1 and Ξ_2 might not be the same, we output the intersection to capture those contexts on which they do agree. The other rules can be found in Figure 3.

3.3 Proof of Correctness and Termination

We prove the system with implicit structural rules sound and complete with respect to the system defined in subsection 3.1.

Theorem 6 (Restricted Weakening is Complete). *A version of context reduction where the added hypothesis may already exist produces the same set of normal form as the one with restricted weakening.*

Proof. One inclusion is immediate. The other permutes contractions with other structural rules, replacing weakening followed by contraction with by mobility.

Theorem 7. *Construction of the set* $\mathsf{NF}_{\Gamma,r}(\Omega) = \{\Omega' \mid \Omega \gg_{\Gamma,r} \Omega'\}$ *terminates.*

Proof. Weakening is bounded by Γ, while contraction is bounded by the number of duplicate variables (which do not increase by restricted weakening).

We need one more lemma to finally prove correctness:

Lemma 2 (Concatenation).
$\forall \Omega \in \mathsf{NF}_{\Gamma,r}(\Omega_1\Omega_2).\, \exists \Omega_1'\Omega_2' \in \mathsf{NF}_{\Gamma,r}(\Omega_1)\mathsf{NF}_{\Gamma,r}(\Omega_2)$ *s.t.* $\Omega_1'\Omega_2' \gg_{\Gamma,r} \Omega$.

Proof. By constructing such Ω_1', Ω_2' from the given derivations.

We now state and prove the correctness theorems.

Theorem 8 (Soundness). *If* $\Gamma \vdash C_r \dashv\!\!\!\dashv \Xi$ *then* $\forall \Omega \in \Xi.\ \Gamma \vdash C_r \dashv \Omega$

Proof. By induction on the derivation in normal form.

Theorem 9 (Completeness).
If $\Gamma \vdash C_r \dashv \Omega$ *then* $\Gamma \vdash C_r \dashv\!\!\!\dashv \Xi$ *for a* Ξ *with* $\mathsf{NF}_{\Gamma,r}(\Omega) \subseteq \Xi$

Proof. By induction on the derivation, with some applications of Lemma 2.

$$\frac{}{\Gamma \vdash 1_m \dashv \mathsf{NF}(\{\cdot\})}\;1I \qquad \frac{\Gamma \vdash 1_m \dashv \Xi_M \quad m \geq r \quad \Gamma \vdash C_r \dashv \Xi_L\, \Xi_R}{\Gamma \vdash C_r \dashv \mathsf{NF}(\Xi_L\, \Xi_M\, \Xi_R)}\;1E$$

$$\frac{\Gamma \vdash A_m \dashv \Xi_L \quad \Gamma \vdash B_m \dashv \Xi_R}{\Gamma \vdash A_m \bullet B_m \dashv \mathsf{NF}(\Xi_L \Xi_R)}\;\bullet I$$

$$\frac{\Gamma \vdash A_m \bullet B_m \dashv \Xi_M \quad m \geq r \quad \Gamma, x{:}A_m, y{:}B_m \vdash C_r \dashv \Xi \quad \Xi \| \Xi_L (x{:}A_m)(y{:}B_m) \Xi_R}{\Gamma \vdash C_r \dashv \mathsf{NF}(\Xi_L \Xi_M \Xi_R)}\;\bullet E$$

$$\frac{\Gamma \vdash A_m \dashv \Xi}{\Gamma \vdash A_m \oplus B_m \dashv \Xi}\;\oplus I_1 \qquad \frac{\Gamma \vdash B_m \dashv \Xi}{\Gamma \vdash A_m \oplus B_m \dashv \Xi}\;\oplus I_2$$

$$\frac{\Gamma \vdash A_m \oplus B_m \dashv \Xi_M \quad m \geq r \quad \begin{array}{c} \Gamma, x{:}A_m \vdash C_r \dashv \Xi_A \\ \Gamma, x{:}B_m \vdash C_r \dashv \Xi_B \end{array} \quad \begin{array}{c} \Xi_A \| \Xi_L^A\,(x{:}A_m)\,\Xi_R^A \\ \Xi_B \| \Xi_L^B\,(x{:}B_m)\,\Xi_R^B \\ \Xi_L \Xi_R = \Xi_L^A \Xi_R^A \cap \Xi_L^B \Xi_R^B \end{array}}{\Gamma \vdash C_r \dashv \mathsf{NF}(\Xi_L \Xi_M \Xi_R)}\;\oplus E$$

$$\frac{\Gamma \vdash A_l \dashv \Xi}{\Gamma \vdash {\downarrow}^l_m A_l \dashv \mathsf{NF}_{\Gamma,m}(\Xi)}\;{\downarrow}I$$

$$\frac{\Gamma \vdash {\downarrow}^l_m A_l \dashv \Xi_M \quad m \geq r \quad \Gamma, x{:}A_l \vdash C_r \dashv \Xi \quad \Xi \| \Xi_L\,(x{:}A_l)\,\Xi_R}{\Gamma \vdash C_r \dashv \mathsf{NF}_{\Gamma,r}(\Xi_L \Xi_M \Xi_R)}\;{\downarrow}E$$

$$\frac{\Gamma, x{:}A_m \vdash B_m \dashv \Xi' \quad \Xi' \| \Xi(x{:}A_m)}{\Gamma \vdash A_m \twoheadrightarrow B_m \dashv \Xi}\;\twoheadrightarrow I \qquad \frac{\Gamma \vdash A_m \twoheadrightarrow B_m \dashv \Xi_L \quad \Gamma \vdash A_m \dashv \Xi_R}{\Gamma \vdash B_m \dashv \mathsf{NF}(\Xi_L \Xi_R)}\;\twoheadrightarrow E$$

$$\frac{\Gamma, x{:}A_m \vdash B_m \dashv \Xi' \quad \Xi' \| (x{:}A_m)\Xi}{\Gamma \vdash A_m \rightarrowtail B_m \dashv \Xi}\;\rightarrowtail I \qquad \frac{\Gamma \vdash A_m \rightarrowtail B_m \dashv \Xi_R \quad \Gamma \vdash A_m \dashv \Xi_L}{\Gamma \vdash B_m \dashv \mathsf{NF}(\Xi_L \Xi_R)}\;\rightarrowtail E$$

$$\frac{\Gamma \vdash A_m \dashv \Xi_A \quad \Gamma \vdash B_m \dashv \Xi_B}{\Gamma \vdash A_m \,\&\, B_m \dashv \Xi_A \cap \Xi_B}\;\&I$$

$$\frac{\Gamma \vdash A_m \,\&\, B_m \dashv \Xi}{\Gamma \vdash A_m \dashv \Xi}\;\&E_1 \qquad \frac{\Gamma \vdash A_m \,\&\, B_m \dashv \Xi}{\Gamma \vdash B_m \dashv \Xi}\;\&E_2$$

$$\frac{\Gamma \vdash A_k \dashv \Xi}{\Gamma \vdash {\uparrow}^m_k A_k \dashv \Xi\|_m}\;{\uparrow}I \qquad \frac{\Gamma \vdash {\uparrow}^m_k A_k \dashv \Xi}{\Gamma \vdash A_k \dashv \mathsf{NF}_{\Gamma,k}(\Xi)}\;{\uparrow}E$$

$$\frac{x{:}A_m \in \Gamma}{\Gamma \vdash A_m \dashv \mathsf{NF}(\{x{:}A_m\})}\;\mathsf{hyp}$$

Fig. 3. Implicit Natural Deduction

4 Discussion and Further Related Work

The closest related work is that of Kanovich et al. [14], who develop a mixed-mode ordered logic using subexponentials. We build on this by providing a sequent calculus with modes uniformly governing structural properties, as in [26, 12], proving cut elimination for this system, and establishing decidability for a natural deduction system with implicit structural rules that corresponds to the initial sequent calculus. The subexponential system of [13] embeds into our framework via $!^m A_{\mathsf{O}} \equiv \downarrow^m_{\mathsf{O}} \uparrow^m_{\mathsf{O}} A_{\mathsf{O}}$ where O is a base mode that admits no structural properties and $m \geq \mathsf{O}$.

The implicit natural deduction system directly yields a type system in which proof terms are independent of structural behavior. For example, both left and right implication share the term $\lambda x.\, e$, and would have the typing judgment $\Gamma \vdash \lambda x.\, e \Longleftarrow A_m \twoheadrightarrow B_m \dashv \Xi$ or $\Gamma \vdash \lambda x.\, e \Longleftarrow A_m \rightarrowtail B_m \dashv \Xi$ in a bidirectional type checker that either computes or verifies an intended type. Note also that all terms remain free of modes.

Directional mobility has been used to model security properties [10] in a semantic rendering of ordered adjoint types in the QRTT language. Our algorithm for computing normal forms can be used to decide *subsumption*, which leads to decidability for a bidirectional type-checker for QRTT. We cannot fully develop this here, just provide a small example. We want to ensure that a high security operation can only be performed after authorization. For this purpose, we introduce three modes: low, auth, and high. A variable $x{:}1_{\mathsf{low}}$ represents a low security operation, a variable $x{:}1_{\mathsf{auth}}$ represents a (successful) authorization, and $x{:}1_{\mathsf{high}}$ represents a high security operation. These modes have the following structural properties: auth is left mobile (authorization can always occur earlier), high is right mobile (we can always perform a high security operation later), and low is mobile in both directions (we can always perform a low security operation). Then sequences such as $(x_1{:}1_{\mathsf{low}})\,(x_2{:}1_{\mathsf{auth}})\,(x_3{:}1_{\mathsf{high}})\,(x_4{:}1_{\mathsf{low}})$ are allowed, but we cannot move x_2 to the right of x_3, or x_3 to the left of x_2.

Future work includes developing and evaluating an efficient type-checking algorithm. Although subsection 3.2 yields a decision procedure, this approach may be of exponential complexity due to unnecessary weakenings and variable placements. Possible optimizations include delaying weakening until a variable exits scope or multiple contexts need to match (e.g., in $\& I$). We could also track fully mobile propositions separately, as they could occur anywhere.

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