

Abstract Syntax

Types	$\tau ::= \tau_1 \rightarrow \tau_2 \mid \forall \alpha. \tau \mid \alpha \mid \tau_1 \times \tau_2 \mid 1 \mid \tau_1 + \tau_2 \mid 0$	
Terms	$e ::= x \mid \lambda x. e \mid e_1 e_2$	$(\rightarrow)$
	$\mid \Lambda \alpha. e \mid e[\tau]$	$(\forall)$
	$\mid \langle e_1, e_2 \rangle \mid \text{case } e (\langle x_1, x_2 \rangle \Rightarrow e')$	$(\times)$
	$\mid \langle \rangle \mid \text{case } e (\langle \rangle \Rightarrow e')$	(1)
	$\mid \mathbf{l} \cdot e \mid \mathbf{r} \cdot e \mid \text{case } e (\mathbf{l} \cdot x_1 \Rightarrow e_1 \mid \mathbf{r} \cdot x_2 \Rightarrow e_2)$	$(+)$
	$\mid \text{case } e ()$	(0)
Contexts	$\Gamma ::= \cdot \mid \Gamma, \alpha \text{ type} \mid \Gamma, x : \tau$	(all variables distinct)

Judgments

$\Gamma \text{ ctx}$	Γ is a valid context	
$\Gamma \vdash \tau \text{ type}$	τ is a valid type	presupposes $\Gamma \text{ ctx}$
$\Gamma \vdash e : \tau$	expression e has type τ	presupposes $\Gamma \text{ ctx}$, ensures $\Gamma \vdash \tau \text{ type}$
$e \text{ value}$	expression e is a value	presupposes $\cdot \vdash e : \tau$ for some τ
$e \mapsto e'$	expression e steps to e'	presupposes $\cdot \vdash e : \tau$ for some τ

Contexts Γ

$\frac{}{(\cdot) \text{ ctx}} \text{ ctx/emp}$	$\frac{\Gamma \text{ ctx}}{(\Gamma, \alpha \text{ type}) \text{ ctx}} \text{ ctx/tpvar}$	$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash \tau \text{ type}}{(\Gamma, x : \tau) \text{ ctx}} \text{ ctx/var}$
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Functions $\tau_1 \rightarrow \tau_2$

$$\frac{\Gamma \vdash \tau_1 \text{ type} \quad \Gamma \vdash \tau_2 \text{ type}}{\Gamma \vdash \tau_1 \rightarrow \tau_2 \text{ type}} \text{ tp/arrow}$$

$$\frac{\Gamma \vdash \tau_1 \text{ type} \quad \Gamma, x_1 : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \lambda x_1 : \tau_1. e_2 : \tau_1 \rightarrow \tau_2} \text{ tp/lam} \quad \frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \text{ tp/var} \quad \frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : \tau_1} \text{ tp/app}$$

$$\frac{}{\lambda x. e \text{ value}} \text{ val/lam} \quad \frac{e_2 \text{ value}}{(\lambda x. e_1) e_2 \mapsto [e_2/x]e_1} \text{ step/app/lam}$$

$$\frac{e_1 \mapsto e'_1}{e_1 e_2 \mapsto e'_1 e_2} \text{ step/app}_1 \quad \frac{e_1 \text{ value} \quad e_2 \mapsto e'_2}{e_1 e_2 \mapsto e_1 e'_2} \text{ step/app}_2$$

Polymorphic Types $\forall\alpha. \tau$

$\frac{\alpha \text{ type} \in \Gamma}{\Gamma \vdash \alpha \text{ type}} \text{ tp/tpvar}$	$\frac{\Gamma, \alpha \text{ type} \vdash \tau \text{ type}}{\Gamma \vdash \forall \alpha. \tau \text{ type}} \text{ tp/forall}$
$\frac{\Gamma, \alpha \text{ type} \vdash e : \tau}{\Gamma \vdash \Lambda \alpha. e : \forall \alpha. \tau} \text{ tp/tplam}$	$\frac{\Gamma \vdash e : \forall \alpha. \tau \quad \Gamma \vdash \sigma \text{ type}}{\Gamma \vdash e [\sigma] : [\sigma / \alpha] \tau} \text{ tp/tpapp}$
$\frac{}{\Lambda \alpha. e \text{ value}} \text{ val/tplam}$	$\frac{}{(\Lambda \alpha. e) [\tau] \mapsto [\tau / \alpha] e} \text{ step/tpapp/tplam}$ $\frac{e \mapsto e'}{e [\tau] \mapsto e' [\tau]} \text{ step/tpapp}$

Pairs $\tau_1 \times \tau_2$

$\frac{\Gamma \vdash \tau_1 \text{ type} \quad \Gamma \vdash \tau_2 \text{ type}}{\Gamma \vdash \tau_1 \times \tau_2 \text{ type}} \text{ tp/prod}$	
$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \langle e_1, e_2 \rangle : \tau_1 \times \tau_2} \text{ tp/pair}$	$\frac{\Gamma \vdash e : \tau_1 \times \tau_2 \quad \Gamma, x_1 : \tau_1, x_2 : \tau_2 \vdash e' : \tau'}{\Gamma \vdash \text{case } e (\langle x_1, x_2 \rangle \Rightarrow e') : \tau'} \text{ tp/casep}$
$\frac{e_1 \text{ value} \quad e_2 \text{ value}}{\langle e_1, e_2 \rangle \text{ value}} \text{ val/pair}$	$\frac{v_1 \text{ value} \quad v_2 \text{ value}}{\text{case } \langle v_1, v_2 \rangle (\langle x_1, x_2 \rangle \Rightarrow e_3) \mapsto [v_1/x_1][v_2/x_2]e_3} \text{ step/casep/pair}$
$\frac{e_1 \mapsto e'_1}{\langle e_1, e_2 \rangle \mapsto \langle e'_1, e_2 \rangle} \text{ step/pair}_1$	$\frac{v_1 \text{ value} \quad e_2 \mapsto e'_2}{\langle v_1, e_2 \rangle \mapsto \langle v_1, e'_2 \rangle} \text{ step/pair}_2$
$\frac{e_0 \mapsto e'_0}{\text{case } e_0 (\langle x_1, x_2 \rangle \Rightarrow e_3) \mapsto \text{case } e'_0 (\langle x_1, x_2 \rangle \Rightarrow e_3)} \text{ step/casep}_0$	

Unit 1

$$\begin{array}{c}
\frac{}{\Gamma \vdash 1 \text{ type}} \text{tp/one} \quad \frac{}{\Gamma \vdash \langle \rangle : 1} \text{tp/unit} \quad \frac{\Gamma \vdash e : 1 \quad \Gamma \vdash e' : \tau'}{\Gamma \vdash \text{case } e (\langle \rangle \Rightarrow e') : \tau'} \text{tp/caseu} \\
\hline
\frac{}{\langle \rangle \text{ value}} \text{val/unit} \quad \frac{}{\text{case } \langle \rangle (\langle \rangle \Rightarrow e) \mapsto e} \text{step/caseu/unit} \\
\frac{e_0 \mapsto e'_0}{\text{case } e_0 (\langle \rangle \Rightarrow e_1) \mapsto \text{case } e'_0 (\langle \rangle \Rightarrow e_1)} \text{step/caseu}_0
\end{array}$$

Sums $\tau_1 + \tau_2$

$$\begin{array}{c}
\frac{\Gamma \vdash \tau_1 \text{ type} \quad \Gamma \vdash \tau_2 \text{ type}}{\Gamma \vdash \tau_1 + \tau_2 \text{ type}} \text{tp/sum} \\
\hline
\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash \tau_2 \text{ type}}{\Gamma \vdash \mathbf{l} \cdot e_1 : \tau_1 + \tau_2} \text{tp/left} \quad \frac{\Gamma \vdash \tau_1 \text{ type} \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \mathbf{r} \cdot e_2 : \tau_1 + \tau_2} \text{tp/right} \\
\frac{\Gamma \vdash e : \tau_1 + \tau_2 \quad \Gamma, x_1 : \tau_1 \vdash e_1 : \sigma \quad \Gamma, x_2 : \tau_2 \vdash e_2 : \sigma}{\Gamma \vdash \text{case } e (\mathbf{l} \cdot x_1 \Rightarrow e_1 \mid \mathbf{r} \cdot x_2 \Rightarrow e_2) : \sigma} \text{tp/cases} \\
\hline
\frac{e_1 \text{ value}}{\mathbf{l} \cdot e_1 \text{ value}} \text{val/left} \quad \frac{e_2 \text{ value}}{\mathbf{r} \cdot e_2 \text{ value}} \text{val/right} \\
\frac{v_1 \text{ value}}{\text{case } \mathbf{l} \cdot v_1 (\mathbf{l} \cdot x_1 \Rightarrow e_1 \mid \mathbf{r} \cdot x_2 \Rightarrow e_2) \mapsto [v_1/x_1]e_1} \text{step/cases/left} \\
\frac{v_2 \text{ value}}{\text{case } \mathbf{r} \cdot v_2 (\mathbf{l} \cdot x_1 \Rightarrow e_1 \mid \mathbf{r} \cdot x_2 \Rightarrow e_2) \mapsto [v_2/x_2]e_2} \text{step/cases/right} \\
\frac{e_1 \mapsto e'_1}{\mathbf{l} \cdot e_1 \mapsto \mathbf{l} \cdot e'_1} \text{step/left} \quad \frac{e_2 \mapsto e'_2}{\mathbf{r} \cdot e_2 \mapsto \mathbf{r} \cdot e'_2} \text{step/right} \\
\frac{e_0 \mapsto e'_0}{\text{case } e_0 (\mathbf{l} \cdot x_1 \Rightarrow e_1 \mid \mathbf{r} \cdot x_2 \Rightarrow e_2) \mapsto \text{case } e'_0 (\mathbf{l} \cdot x_1 \Rightarrow e_1 \mid \mathbf{r} \cdot x_2 \Rightarrow e_2)} \text{step/cases}_0
\end{array}$$

Empty Type 0

$\frac{}{\Gamma \vdash 0 \text{ type}}$ tp/zero	(no constructor)	$\frac{\Gamma \vdash e_0 : 0 \quad \Gamma \vdash \tau \text{ type}}{\Gamma \vdash \text{case } e_0 () : \tau}$ tp/casez
(no values)	$\frac{e_0 \mapsto e'_0}{\text{case } e_0 () \mapsto \text{case } e'_0 ()}$ step/casez ₀	

Theorems

Preservation. If $\cdot \vdash e : \tau$ and $e \mapsto e'$ then $\cdot \vdash e' : \tau$.

Progress. For every expression $\cdot \vdash e : \tau$ either $e \mapsto e'$ for some e' or e *value*.

Finality of Values. There is no $\cdot \vdash e : \tau$ such that $e \mapsto e'$ for some e' and e *value*.

Sequentiality. If $e \mapsto e_1$ and $e \mapsto e_2$ then $e_1 = e_2$.

Canonical Forms. Assume $\cdot \vdash v : \tau$ and v *value*.

- (i) If $\tau = \tau_1 \rightarrow \tau_2$ then $v = \lambda x. e_2$ for some e_2
- (ii) If $\tau = \forall \alpha. \tau'$ then $v = \Lambda \alpha. e'$ for some e'
- (iii) If $\tau = \tau_1 \times \tau_2$ then $v = \langle v_1, v_2 \rangle$ for some v_1 *value* and v_2 *value*
- (iv) If $\tau = 1$ then $v = \langle \rangle$
- (v) If $\tau = \tau_1 + \tau_2$ then $v = \mathbf{l} \cdot v_1$ for some v_1 *value* or $v = \mathbf{r} \cdot v_2$ for some v_2 *value*
- (vi) If $\tau = 0$ then we have a contradiction