Recitation 4: Verifications and Quantifiers

Jon Sterling

“Proof search” is not a mere matter of practice: it is *praxis*. The dialectic of proof search is to discover ways to pare down the state space of a logic, and then synthesize this into a new logic which is exactly as expressive as the old one. This new restricted logic not only has better search complexity, but also exposes critical semantic content which tend to have been obscured in the original logic.

Perhaps the most famous example of this process is Andreoli’s *focalization*; in recent lectures, we have begun to study a simpler instance of this process, namely the decomposition of truth into *verification* and *use*. The passage to verifications constitutes a collation of upward and downward deductions respectively.

1 Verifications and Uses

“Verifications” are proofs that proceed upwards from conclusions to premises; this is also known as *backward inference* or *refinement-style proof*. On the other hand, “uses” are proofs that proceed from premise to conclusion, also known as *forward inference*. The judgment $A\uparrow$ stands for verifications of A, and the judgment $A\downarrow$ stands for uses of A.

The rules for verifications and uses of the conjunction connective are as follows:

$$\frac{A\uparrow \quad B\uparrow}{A\wedge B\uparrow}\,\wedge\mathsf{I} \qquad\qquad \frac{A\wedge B\downarrow}{A\downarrow}\,\wedge\mathsf{E}_1 \qquad\qquad \frac{A\wedge B\downarrow}{B\downarrow}\,\wedge\mathsf{E}_2$$

On this basis, you may think that verifications correspond to introduction forms and uses correspond to elimination forms. This is not correct, as can be seen from the case of disjunction:

$$\frac{A\uparrow}{A\vee B\uparrow}\,\vee\mathsf{I}_1 \qquad\qquad \frac{B\uparrow}{A\vee B\uparrow}\,\vee\mathsf{I}_2 \qquad\qquad \frac{A\vee B\downarrow \quad \begin{array}{c}\overline{A\downarrow}\,u\\ \vdots\\ C\uparrow\end{array} \quad \begin{array}{c}\overline{B\downarrow}\,v\\ \vdots\\ C\uparrow\end{array}}{C\uparrow}\,\vee\mathsf{E}^{u,v}$$

Question. *Will the elimination rule for implication result have a verification or a use in its conclusion?*

$$\dfrac{\begin{array}{c}\overline{A\downarrow}^{\,u}\\ \vdots\\ B\uparrow\end{array}}{A\supset B\uparrow}\supset\!\mathsf{I}^{u}\qquad\qquad \dfrac{A\supset B\downarrow\quad A\uparrow}{B\downarrow}\supset\!\mathsf{E}$$

Remark (Bonus)**.** *One dimension along which connectives vary is* polarity*: some connectives are positive, and some are negative. We cannot yet make this distinction precise, but some students have already begun to observe it. Later on, we may see that negative connectives have elimination forms as uses, but positive connectives have elimination forms as verifications.*

The calculus of verifications and uses has one extra rule which was not visible in the original logic:

$$\dfrac{A\downarrow}{A\uparrow}\updownarrow$$

Question. *Would it be reasonable to add the inverse of the above rule, which concludes* $A\downarrow$ *from* $A\uparrow$*? What would be the consequences of this?*

We have also begun to study quantifiers (universal and existential). The rules for these are as follows:

$$\dfrac{\begin{array}{c}\overline{c : A}\\ \vdots\\ A(c)\uparrow\end{array}}{\forall x:\tau.\,A(x)\uparrow}\forall\mathsf{I}^{c}\qquad\qquad \dfrac{\forall x:\tau.\,A(x)\downarrow\quad t:\tau}{A(t)\downarrow}\forall\mathsf{E}$$

$$\dfrac{t:\tau\quad A(t)\uparrow}{\exists x:\tau.\,A(x)\uparrow}\exists\mathsf{I}\qquad\qquad \dfrac{\exists x:\tau.\,A(x)\downarrow\qquad\begin{array}{c}\overline{c:\tau}\quad\overline{A(c)\downarrow}^{\,u}\\ \vdots\\ C\uparrow\end{array}}{C\uparrow}\exists\mathsf{E}^{c,u}$$

2 Examples with quantifiers

Consider a predicate $A(x)$ which depends on $x : \tau$ and a proposition B.

2.1 Existential Adjointness

Prove the following equivalence in the logic of verifications and uses:

$$(\forall x:\tau.\,(A(x)\supset B))\equiv((\exists x:\tau.\,A(x))\supset B)\uparrow$$

Remark. *For the advanced, the above exercise is essentially the fact that the existential quantifier is "left adjoint" to weakening.*

Proof. First the proof from left to right:

$$\dfrac{\dfrac{\dfrac{\overline{\exists x:\tau.\,A(x)\downarrow}^{\,v} \qquad \dfrac{\dfrac{\dfrac{\overline{\forall x:\tau.\,(A(x)\supset B)\downarrow}^{\,u} \quad \overline{c:\tau}}{A(c)\supset B\downarrow}\,\forall\mathsf{E} \qquad \dfrac{\overline{A(c)\downarrow}^{\,w}}{A(c)\uparrow}\,\updownarrow}{B\downarrow}\,{\supset}\mathsf{E}}{B\uparrow}\,\updownarrow}{B\uparrow}\,\exists\mathsf{E}^{c,w}}{(\exists x:\tau.\,A(x))\supset B\uparrow}\,{\supset}\mathsf{I}^{v}}{(\forall x:\tau.\,(A(x)\supset B))\supset((\exists x:\tau.\,A(x))\supset B)\uparrow}\,{\supset}\mathsf{I}^{u} \tag*{$(\Rightarrow)$}$$

Next, the proof from right to left:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{(\exists x:\tau.\,A(x))\supset B\downarrow}^{\,u} \qquad \dfrac{\overline{c:\tau} \quad \dfrac{\overline{A(c)\downarrow}^{\,v}}{A(c)\uparrow}\,\updownarrow}{\exists x:\tau.\,A(x)\uparrow}\,\exists\mathsf{I}}{B\downarrow}\,{\supset}\mathsf{E}}{B\uparrow}\,\updownarrow}{A(c)\supset B\uparrow}\,{\supset}\mathsf{I}^{v}}{\forall x:\tau.\,(A(x)\supset B)\uparrow}\,\forall\mathsf{I}^{c}}{((\exists x:\tau.\,A(x))\supset B)\supset(\forall x:\tau.\,(A(x)\supset B))\uparrow}\,{\supset}\mathsf{I}^{u} \tag*{$(\Leftarrow)$}$$

□

2.2 Universal Adjointness

Prove the following equivalence in the logic of verifications and uses:

$$(\forall x:\tau.\,(B\supset A(x)))\equiv(B\supset\forall x:\tau.\,A(x))\uparrow$$

Remark. *Likewise, this is the fact that the universal quantifier is "right adjoint" to weakening.*

Proof. First the proof from left to right:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\forall x:\tau.\,(B\supset A(x))\downarrow}^{\,u} \quad \overline{c:\tau}}{B\supset A(c)\downarrow}\,\forall\mathsf{E} \qquad \dfrac{\overline{B\downarrow}^{\,v}}{B\uparrow}\,\updownarrow}{A(c)\downarrow}\,{\supset}\mathsf{E}}{A(c)\uparrow}\,\updownarrow}{\forall x:\tau.\,A(x)\uparrow}\,\forall\mathsf{I}^{c}}{B\supset\forall x:\tau.\,A(x)\uparrow}\,{\supset}\mathsf{I}^{v}}{(\forall x:\tau.\,(B\supset A(x)))\supset(B\supset\forall x:\tau.\,A(x))\uparrow}\,{\supset}\mathsf{I}^{u} \tag*{$(\Rightarrow)$}$$

Next, the proof from right to left:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{B \supset \forall x:\tau.\,A(x)\downarrow}\;u \qquad \dfrac{\overline{B\downarrow}\;v}{B\uparrow}\updownarrow}{\forall x:\tau.\,A(x)\downarrow}\supset\!\mathsf{E} \qquad \overline{c:\tau}}{A(c)\downarrow}\forall\mathsf{E}}{A(c)\uparrow}\updownarrow}{B\supset A(c)\uparrow}\supset\!\mathsf{I}^v}{\forall x:\tau.\,(B\supset A(x))\uparrow}\forall\mathsf{I}^c}{(B\supset \forall x:\tau.\,A(x))\supset(\forall x:\tau.\,(B\supset A(x)))\uparrow}\supset\!\mathsf{I}^u \qquad (\Leftarrow)$$

□

2.3 Swapping Quantifiers

If there is time, try proving that an existential quantification can be moved underneath a universal quantification. Fixing a predicate in two variables $A(x, y)$ for $x : \sigma, y : \tau$:

$$\dfrac{\dfrac{\dfrac{\overline{\exists y:\tau.\,\forall x:\sigma.\,A(x,y)\downarrow}\;u \qquad \dfrac{\overline{d:\tau} \qquad \dfrac{\dfrac{\overline{\forall x:\sigma.\,A(x,d)\downarrow}\;v \qquad \overline{c:\sigma}}{A(c,d)\downarrow}\forall\mathsf{E}}{A(c,d)\uparrow}\updownarrow}{\exists y:\tau.\,A(c,y)\uparrow}\exists\mathsf{I}}{\exists y:\tau.\,A(c,y)\uparrow}\exists\mathsf{E}^{d,v}}{\forall x:\sigma.\,\exists y:\tau.\,A(x,y)\uparrow}\forall\mathsf{I}^c}{(\exists y:\tau.\,\forall x:\sigma.\,A(x,y))\supset(\forall x:\sigma.\,\exists y:\tau.\,A(x,y))\uparrow}\supset\!\mathsf{I}^u$$