Algorithms: Solutions 4

Problem 1
For each of the following functions, give an asymptotically tight bound.

(a) $(n+2)\cdot(n+3)\cdot(n+6) = \Theta(n)\cdot\Theta(n)\cdot\Theta(n) = \Theta(n\cdot n\cdot n) = \Theta(n^3)$

(b) $(n+2)^2\cdot(n+3)^3\cdot(n+6)^6 = (\Theta(n))^2\cdot(\Theta(n))^3\cdot(\Theta(n))^6 = \Theta(n^2\cdot n^3\cdot n^6) = \Theta(n^{11})$

(c) $\sqrt{2n+2}\cdot\sqrt[3]{3n+3}\cdot\sqrt[6]{6n+6} = \Theta(n^{1/2}\cdot n^{1/3}\cdot n^{1/6}) = \Theta(n^{1/2+1/3+1/6}) = \Theta(n)$

(d) $2^{6\cdot n}+6^{2\cdot n} = (2^6)^n+(6^2)^n = 64^n+36^n = 64^n+o(64^n) = \Theta(64^n)$

(e) $(\sqrt{n})^n+n^{\sqrt{n}} = (n^{1/2})^n+n^{\sqrt{n}} = n^{n/2}+o(n^{n/2}) = \Theta(n^{n/2})$

(f) $2^{\left(2^{\lg\left(\frac{\log_3 n}{\log_3 2}\right)}\right)} = 2^{\left(2^{\lg(\lg n)}\right)} = 2^{\lg n} = n = \Theta(n)$

Problem 2
Give an example of functions $f(n)$ and $g(n)$ that satisfy the following conditions:

$$f(n) = O(g(n))$$
$$f(n) \neq \Theta(g(n))$$
$$f(n) \neq o(g(n))$$

Consider the following two functions:

$$\begin{aligned} f(n) &= 1 \\ g(n) &= \begin{cases} 1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Since $f(n) \leq g(n)$, we immediately conclude that $f(n) = O(g(n))$. For even n, $f(n)$ is of the same order as $g(n)$, which means that $f(n) \neq o(g(n))$. On the other hand, for odd n, $f(n)$ grows asymptotically slower than $g(n)$, which implies that $f(n) \neq \Theta(g(n))$.