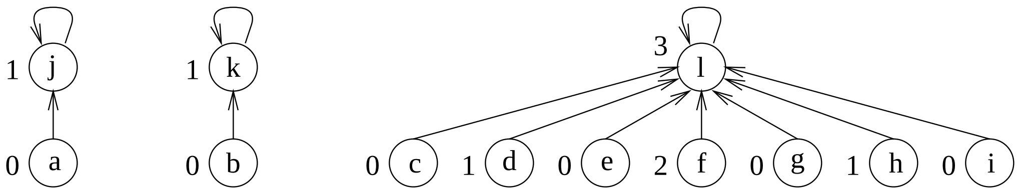
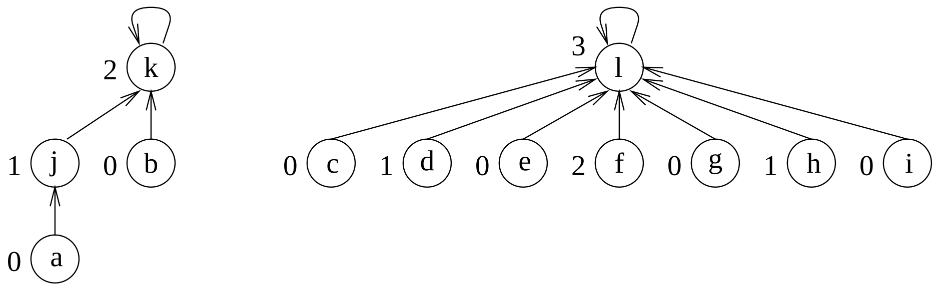
Analysis of Algorithms: Solutions 7

```
                                      X
                                      X
                                      X   X
                                  X   X   X
                              X   X   X   X
number of                     X   X   X   X
homeworks                     X   X   X   X
                              X   X   X   X
                              X   X   X   X
                              X   X   X   X
              X           X   X   X   X   X
              X   X       X   X   X   X   X
              ------------------------------
              3   4   5   6   7   8   9   10
                        grades
```

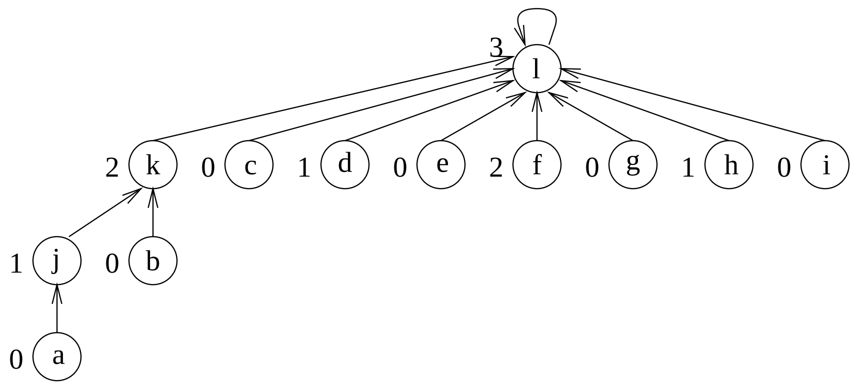
Problem 1
Consider the disjoint-set forest below, where numbers are the ranks of elements, and suppose that you apply three successive operations: UNION(a, b), UNION(b, c), and FIND-SET(a). Give a picture of the disjoint forest after each of these operations.



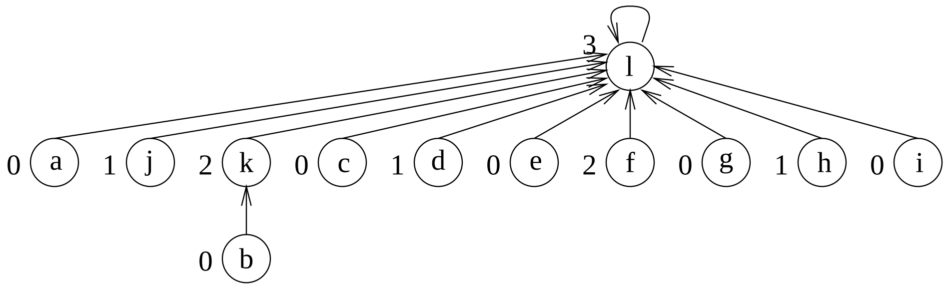
UNION(a, b):



UNION(b, c):



FIND-SET(a):



Problem 2
Write pseudocode for MAKE-SET, FIND-SET, and UNION, using the linked-list representation of disjoint sets.

We use four fields for each element x of a linked list:

> $next[x]$: pointer to the next element of the list; NIL if x is the last element
> $rep[x]$: pointer to the set representative, that is, to the first element of the list
> $last[x]$: if x is the first element of a list, then this field points to the last element
> $size[x]$: if x is the first element, then this field contains the size of the list

If x is not the first element of a list, then the algorithms do *not* use its *last* and *size* fields, and the information in these fields may be incorrect.

MAKE-SET(x)
$next[x] \leftarrow$ NIL
$rep[x] \leftarrow x$
$last[x] \leftarrow x$
$size[x] \leftarrow 1$

FIND-SET(x)
return $rep[x]$

UNION(x, y)
if $size[rep[x]] > size[rep[y]]$
    then APPEND($rep[x], rep[y]$)
    else APPEND($rep[y], rep[x]$)

APPEND(x, y)
$next[last[x]] \leftarrow y$
$size[x] \leftarrow size[x] + size[y]$
$z \leftarrow y$
while $z \neq$ NIL $\triangleright$ change the *rep* pointers in the second list
    do $rep[z] \leftarrow x$
        $z \leftarrow next[z]$

Problem 3

Suppose that $A[1..n]$ and $B[1..m]$ are sorted arrays, and $n \leq m$. Write an algorithm that finds their smallest common element; if they have no common elements, it should return 0.

The intuitive idea is to divide $B[1..m]$ into segments, each of size $k = m/n$, and perform binary search in each segment. We need to use a version of binary search, BIN-SEARCH(B, p, r, k), which searches for an element k in a segment $B[p..r]$. If this version finds k, it returns the corresponding index of B; if not, it returns the index of the next larger element. For example, if $k = 6$ and $B[p..r] = \langle 3, 5, 7, 9 \rangle$, the search returns the index of 7. The following algorithm calls BIN-SEARCH on k-element segments of B.

```
COMMON-ELEMENT(A, B, n, m)
k ← ⌊m/n⌋
i ← 1
j ← 1
while i ≤ n and j ≤ m
    do  if A[i] = B[j]
            then return A[i]
        if A[i] < B[j]
            then i = i + 1
            else repeat j = j + k
                     until j > m or A[i] ≤ B[j]
                 j ← BIN-SEARCH(B, j − k + 1, min(j, m), A[i])
return 0
```

The running time of COMMON-ELEMENT is $O(n \cdot (1 + \lg \frac{m}{n}))$. In particular, if A and B are of about the same size, then the time is $O(m)$. On the other hand, if A is much smaller than B, the running time is significantly better than $O(m)$.