Analysis of Algorithms: Solutions 1

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number of            X
homeworks            X
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               X  X  X
               X  X  X
         X  X  X  X  X
      X  X  X  X  X  X
      -----------------
      5  6  7  8  9  10
           grades
```

The histogram shows the distribution of grades, from 0 to 10.

Problem 1

Let $A[1..n]$ be a sorted array of n distinct numbers. Write an efficient algorithm BINARY-SEARCH(A, n, k) that finds a given value k in the array $A[1..n]$. It should return the index of the found element; if the array does not include k, the algorithm should return 0.

BINARY-SEARCH(A, n, k)
$p \leftarrow 1$
$r \leftarrow n$
while $p < r$
  do $q = \lfloor (p+r)/2 \rfloor$
    if $k \leq A[q]$
      then $r \leftarrow q$
      else $p \leftarrow q + 1$
if $k = A[p]$
  then return p
  else return 0

Problem 2

(a) $1 + 2 + 3 + 4 + ... + n = \frac{n \cdot (n+1)}{2}$.

We use a proof by induction. Clearly, the equality holds for $n = 1$; we now show that, if it holds for n, then it also holds for $n + 1$:

$$\begin{aligned}
1 + 2 + ... + n + (n+1) &= (1 + 2 + ... + n) + (n+1) \\
&= \frac{n \cdot (n+1)}{2} + (n+1) \\
&= \frac{n \cdot (n+1) + 2 \cdot (n+1)}{2} \\
&= \frac{(n+2) \cdot (n+1)}{2} \\
&= \frac{(n+1) \cdot ((n+1)+1)}{2}
\end{aligned}$$

(b) $1 + x + x^2 + x^3 + ... + x^n = \frac{x^{n+1}-1}{x-1}$ (where $x \neq 1$).

We observe that the equality holds for $n = 1$, and apply induction to "step" from n to $n+1$:

$$\begin{aligned}
1 + x + ... + x^n + x^{n+1} &= \frac{x^{n+1} - 1}{x - 1} + x^{n+1} \\
&= \frac{x^{n+1} - 1 + x^{n+1} \cdot (x-1)}{x - 1} \\
&= \frac{x^{n+1} - 1 + x^{n+2} - x^{n+1}}{x - 1} \\
&= \frac{x^{(n+1)+1} - 1}{x - 1}
\end{aligned}$$