10-701 Machine Learning, Fall 2011: Homework 4

Due 11/16 at the beginning of class.

November 2, 2011

1 Bayesian network inference [Suyash, 25 points]

1.1 Variable elimination [12 points]

Consider the Bayesian network shown in Figure 1. Suppose we want to compute the marginal for node H, i.e, Probability(H=h). We will examine the effect of order of variable elimination on the amount of computation required. Assume each variable can take only two values.

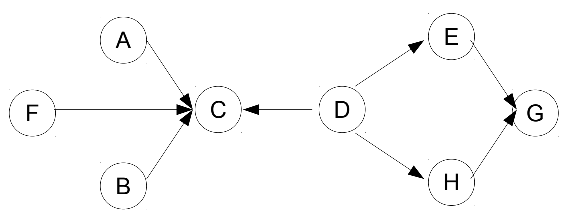


Figure 1: Bayesian network for variable elimination

- [**5 points**] Consider the elimination order: A,B,F,C,D,E,G. Write down the factors that you will encounter during this elimination. How many additions does it require?
- [**5 points**] Consider the elimination order: C,D,F,A,B,E,G. Write down the factors that you will encounter during this elimination. How many additions does it require?
- [**2 points**] Which elimination order is better in terms of computation and storage?

1.2 Graph construction [9 points]

1. Suppose there is Bayesian network with n vertices $X_1, \cdots, X_n$. We want to compute $P(X_2 = x)$. Your task is to draw the edges in the Bayesian network so that the graph has the following two properties:

   - [**2 points**] There exists an elimination order such that computing $P(X_2 = x)$ takes time linear in n.
   - [**2 points**] There exists an elimination order such that computing $P(X_2 = x)$ takes time exponential in n.

   Draw the graph ([**3 points**]) and write the two elimination orders we desire. Note that your set of edges should be the same for both cases.

2. [**2 points**] State true or false with brief explanation: If we have learnt a Naive Bayes classifier with n features and a class label, all of which are boolean, then for computing $P(X_1 = 0)$ there exists an order of elimination of the other variables that takes exponential time in terms of n.

1.3 Learning Bayes Nets [4 points]

Suppose you want to learn a Bayes net over two binary variables X_1 and X_2. You have N training pairs of X_1 and X_2, given as $\{(x_1^{(1)}, x_2^{(1)}), \cdots, (x_1^{(N)}, x_2^{(N)})\}$. For any Bayes net you learn its parameters using maximum likelihood estimation. Let A denote the BN with no edges, and B denote the BN with an edge from X_1 to X_2.

- [**2 points**] Describe a case when choosing B to model the data is better than choosing A.
- [**2 points**] Describe a case when choosing A to model the data is better than choosing B. (Hint: Your choice may be determined by factors other than how well the BN fits the data).

2 Undirected Graphical Models [25pt, Nan Li]

A Markov Random Field is an undirected graphical model. Unlike Bayesian Networks, undirected edges in the graph simply give correlations between variables. Figure 2 is a Markov Random Field. Assume that all the variables are boolean. Each edge in the graph, $\forall i, j \in \{1, 2, 3, 4\}, i \neq j, e_{ij} = \langle x_i, x_j \rangle$, corresponds to a potential $\Psi_{e_{ij}}(x_i, x_j)$. We will explore the probability distribution encoded in this graph in this problem.

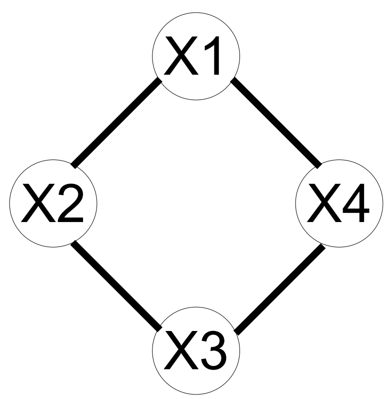


Figure 2: A Markov Random Field

2.1 Representation [4 pt]

(a) [2 pt] Please write down the formula to calculate the joint probability of all variables defined by the above Markov Random Field in terms of the given potentials.

(b) [2 pt] How many parameters do we need to store the potentials in the graph?

2.2 Independence [5 pt]

(a) [2 pt] What is the Markov blanket of variable x_2?

(b) [3 pt] Is $x_1 \perp x_3 | x_2, x_4$? Please answer yes or no, and briefly explain why.

2.3 Hammersley-Clifford Theorem [8 pt]

Now let's consider a probability distribution P over x_1, x_2, x_3, x_4, which gives probability 1/8 to each of the following configurations: $(0,0,0,0)$, $(1,0,0,0)$, $(1,1,0,0)$, $(1,1,1,0)$, $(0,0,0,1)$, $(0,0,1,1)$, $(0,1,1,1)$, $(1,1,1,1)$. All other configurations are given probability zero.

(a) [3 pt] Is $x_1 \perp x_3 | x_2, x_4$ true in the above distribution? Please answer yes or no, and briefly explain why.

(b) [5 pt] The MRF shown in Figure 2 is actually an I-map for P. Based on the Hammersley-Clifford Theorem taught in class, does this mean that the distribution P factorizes according to the given Markov Random Field, i.e. can you find a set of potential values for $\Psi_{e_{ij}(x_i,x_j)}$ so that it defines the distribution P? Please answer yes or no. If yes, please show the potentials. If no, please prove it.

2.4 Partition Function [8 pt]

To define a probability distribution in an MRF, we need a need a normalization factor Z known as the partition function. Can we simply renormalize the values in each potential function so that they sum up to one, and then get rid of the normalization factor Z?

(a) [4 pt] If we scale only one potential function by a positive constant, and compute a new Z accordingly, how does it affect the distribution defined by the Markov network? Please prove your answer.

(b) [4 pt] If we scale all factors to sum to 1 locally, does that imply $Z = 1$? If yes, please prove it. If no, please give a counter example.

3 SVMs [Qirong Ho, 25 points]

3.1 Feature Mappings

Suppose you are given 6 one-dimensional points: 3 with negative labels $x_1 = -1$, $x_2 = 0$, $x_3 = 1$ and 3 with positive labels $x_4 = -2$, $x_5 = 2$, $x_6 = 3$.

1. **[1 point]** Draw the 6 points on the one-dimensional line, using circles to represent the positive labels and squares to represent the negative labels.

2. **[2 points]** The 6 points are not linearly separable. Write down a feature transformation $f(x)$ such that the transformed points $f(x_1), f(x_2), \ldots, f(x_6)$ are linearly separable.

3. **[2 points]** Draw your 6 *transformed* points from part 2 on the one-dimensional line. Then, draw the decision boundary given by the hard-margin linear SVM, and indicate which points are support vectors.

4. **[2 points]** Your decision boundary from part 3 has the form $w_0 + w_1 f(x)$. Give the values of w_0 and w_1.

5. **[2 points]** Now suppose we transform the 6 points to the feature space $(x, f(x))$, where $f(x)$ is your feature transformation from part 2. In other words, you now have 6 two-dimensional points $(x_1, f(x_1)), (x_2, f(x_2)), \ldots, (x_6, f(x_6))$. Draw these 6 points on the two-dimensional plane, along with the decision boundary given by the hard-margin linear SVM. Finally, indicate which points are support vectors.

6. **[2 points]** Your decision boundary from part 5 has the form $w_0 + w_1 x + x_2 f(x)$. Give the values of w_0, w_1, w_2.

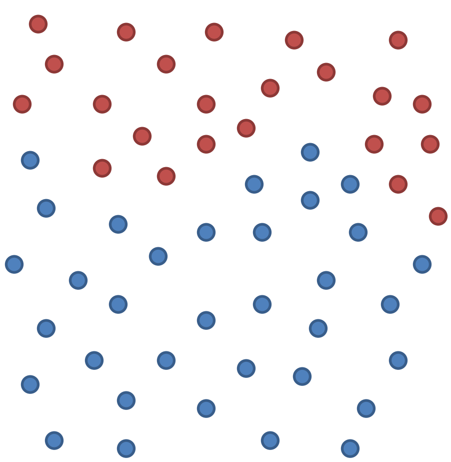
7. **[2 points]** The feature mapping $x \longmapsto (x, f(x))$ in parts 5 and 6 is associated with a kernel $K(x, x')$ where x, x' are points in the original one-dimensional feature space. Write down this kernel.

8. **[3 points]** What is the VC dimension of the hard-margin linear SVM in the feature space $(x, f(x))$? That is to say, what is the largest number of points in $(x, f(x))$ that can be shattered by a linear classifier?

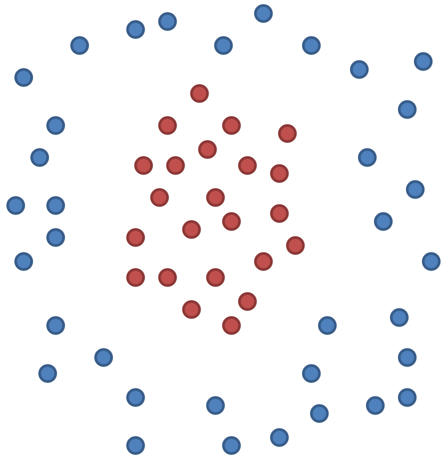
3.2 Kernels

For each of the figures below, give a kernel that allows the hard-margin linear SVM to classify the red and blue points perfectly. No need to write down the mathematical form of the kernel, just state its type.

1. **[3 points]**



2. **[3 points]**



3. **[3 points]**