CMU SCS

15-826: Multimedia Databases and Data Mining

Lecture #20: SVD - part III (more case studies)

C. Faloutsos

CMU SCS

Must-read Material

- Textbook Appendix D
- Kleinberg, J. (1998). Authoritative sources in a hyperlinked environment. Proc. 9th ACM-SIAM Symposium on Discrete Algorithms.
- Brin, S. and L. Page (1998). Anatomy of a Large-Scale Hypertextual Web Search Engine. 7th Intl World Wide Web Conf.

15-826 2

CMU SCS

Outline

Goal: 'Find similar / interesting things'

- Intro to DB
- Indexing - similarity search
- Data Mining

15-826 3

CMU SCS

Indexing - Detailed outline

- primary key indexing
- secondary key / multi-key indexing
- spatial access methods
- fractals
- text
- Singular Value Decomposition (SVD)
- multimedia
- ...

CMU SCS

SVD - Detailed outline

- Motivation
- Definition - properties
- Interpretation
- Complexity
- Case studies
- SVD properties
- More case studies
- Conclusions

CMU SCS

SVD - detailed outline

- ...
- Case studies
- SVD properties
- more case studies
  - google/Kleinberg algorithms
  - query feedbacks
- Conclusions

CMU SCS

SVD - Other properties - summary

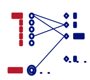
- can produce orthogonal basis (obvious) (who cares?)
- can solve over- and under-determined linear problems (see C(1) property)
- can compute 'fixed points' (= 'steady state prob. in Markov chains') (see C(4) property)

CMU SCS

IMPORTANT!

Properties – sneak preview:

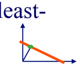
A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \mathbf{\Lambda}_{[r \times r]} \mathbf{V}^{\mathbf{T}}_{[r \times m]}$



B(5): $(\mathbf{A}^{T} \mathbf{A})^{k}\ \mathbf{v'} \sim (\text{constant})\ \mathbf{v}_1$

C(1): $\mathbf{A}_{[n \times m]}\ \mathbf{x}_{[m \times 1]} = \mathbf{b}_{[n \times 1]}$

then, $\mathbf{x}_0 = \mathbf{V}\ \mathbf{\Lambda}^{(-1)}\ \mathbf{U}^{T}\ \mathbf{b}$: shortest, actual or least-squares solution



C(4): $\mathbf{A}^{\mathbf{T}}\ \mathbf{A}\ \mathbf{v}_1 = \lambda_1^2\ \mathbf{v}_1$

CMU SCS

SVD -outline of properties

- (A): obvious
- (B): less obvious
- (C): least obvious (and most powerful!)

CMU SCS

Properties - by defn.:

A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \mathbf{\Lambda}_{[r \times r]} \mathbf{V^T}_{[r \times m]}$

A(1): $\mathbf{U}^T_{[r \times n]} \mathbf{U}_{[n \times r]} = \mathbf{I}_{[r \times r]}$ (identity matrix)
A(2): $\mathbf{V}^T_{[r \times n]} \mathbf{V}_{[n \times r]} = \mathbf{I}_{[r \times r]}$
A(3): $\mathbf{\Lambda}^k = \text{diag}(\lambda_1^k, \lambda_2^k, \ldots \lambda_r^k)$ (k: ANY real number)
A(4): $\mathbf{A^T} = \mathbf{V} \mathbf{\Lambda} \mathbf{U^T}$

15-826 Copyright: C. Faloutsos (2012) 10

CMU SCS

Less obvious properties

A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \mathbf{\Lambda}_{[r \times r]} \mathbf{V^T}_{[r \times m]}$

B(1): $\mathbf{A}_{[n \times m]} (\mathbf{A}^T)_{[m \times n]} = ??$

15-826 Copyright: C. Faloutsos (2012) 11

CMU SCS

Less obvious properties

A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \mathbf{\Lambda}_{[r \times r]} \mathbf{V^T}_{[r \times m]}$
B(1): $\mathbf{A}_{[n \times m]} (\mathbf{A}^T)_{[m \times n]} = \mathbf{U} \mathbf{\Lambda}^2 \mathbf{U}^T$
symmetric; Intuition?

15-826 Copyright: C. Faloutsos (2012) 12

CMU SCS

Less obvious properties

A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \mathbf{\Lambda}_{[r \times r]} \mathbf{V^T}_{[r \times m]}$

B(1): $\mathbf{A}_{[n \times m]} (\mathbf{A}^T)_{[m \times n]} = \mathbf{U} \mathbf{\Lambda}^2 \mathbf{U}^T$

symmetric; Intuition?

'document-to-document' similarity matrix

B(2): symmetrically, for '**V**'

$(\mathbf{A}^T)_{[m \times n]} \mathbf{A}_{[n \times m]} = \mathbf{V} \mathbf{\Lambda}^2 \mathbf{V}^T$

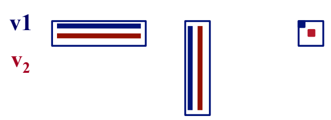
Intuition?

15-826 Copyright: C. Faloutsos (2012) 13

CMU SCS

Reminder: 'column orthonormal'

- $\mathbf{V}^T$ $\mathbf{V}$ = $\mathbf{I}_{[r \times r]}$



$\mathbf{v_1}^T \times \mathbf{v_1} = 1$

$\mathbf{v_1}^T \times \mathbf{v_2} = 0$

15-826 Copyright: C. Faloutsos (2012) 14

CMU SCS

Less obvious properties

A: term-to-term similarity matrix

B(3): $((\mathbf{A}^T)_{[m \times n]} \mathbf{A}_{[n \times m]})^k = \mathbf{V} \mathbf{\Lambda}^{2k} \mathbf{V}^T$

and

B(4): $(\mathbf{A}^T \mathbf{A})^k \sim \mathbf{v}_1 \lambda_1^{2k} \mathbf{v}_1^T$ for k>>1

where

$\mathbf{v}_1$: [m x 1] first column (singular-vector) of **V**

λ_1: strongest singular value

15-826 Copyright: C. Faloutsos (2012) 15

CMU SCS

Proof of (B4)?

CMU SCS

Less obvious properties

B(4): $(\mathbf{A}^T \mathbf{A})^k \sim \mathbf{v}_1 \lambda_1^{2k} \mathbf{v}_1^T$ for k>>1

B(5): $(\mathbf{A}^T \mathbf{A})^k \mathbf{v'} \sim (\text{constant})\ \mathbf{v}_1$

ie., for (almost) any **v'**, it converges to a vector parallel to $\mathbf{v}_1$

Thus, useful to compute first singular vector/ value (as well as the next ones, too...)

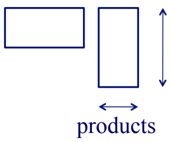
CMU SCS

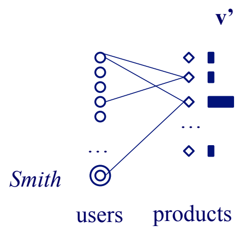
Proof of (B5)?

CMU SCS

Property (B5)

- Intuition:
  - $(A^T A)\ v'$
  - $(A^T A)^k\ v'$



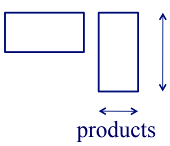


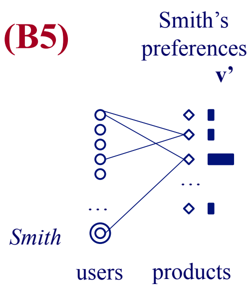
15-826 Copyright: C. Faloutsos (2012) 19

CMU SCS

Property (B5)

- Intuition:
  - $(A^T A)\ v'$
  - $(A^T A)^k\ v'$



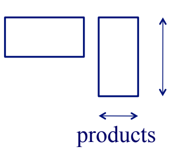


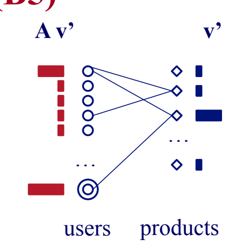
15-826 Copyright: C. Faloutsos (2012) 20

CMU SCS

Property (B5)

- Intuition:
  - $(A^T A)\ v'$
  - $(A^T A)^k\ v'$



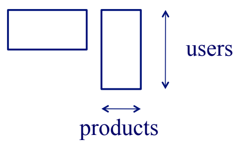


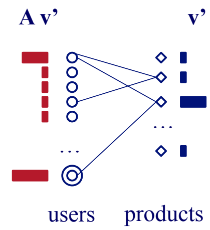
15-826 Copyright: C. Faloutsos (2012) 21

CMU SCS

Property (B5)

- Intuition:
  - $(A^T A)\ v'$
  - $(A^T A)^k\ v'$

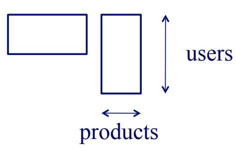


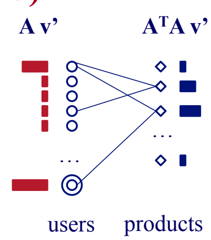


CMU SCS

Property (B5)

- Intuition:
  - $(A^T A)\ v'$
  - $(A^T A)^k\ v'$





CMU SCS

Property (B5)

- Intuition:
  - $(A^T A)\ v'$ what Smith's 'friends' like
  - $(A^T A)^k\ v'$ what k-step-away-friends like

(ie., after *k* steps, we get what everybody likes, and Smith's initial opinions don't count)

CMU SCS

Less obvious properties - repeated:

A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \mathbf{\Lambda}_{[r \times r]} \mathbf{V^T}_{[r \times m]}$

B(1): $\mathbf{A}_{[n \times m]} (\mathbf{A}^T)_{[m \times n]} = \mathbf{U} \mathbf{\Lambda}^2 \mathbf{U}^T$

B(2): $(\mathbf{A}^T)_{[m \times n]} \mathbf{A}_{[n \times m]} = \mathbf{V} \mathbf{\Lambda}^2 \mathbf{V}^T$

B(3): $((\mathbf{A}^T)_{[m \times n]} \mathbf{A}_{[n \times m]})^k = \mathbf{V} \mathbf{\Lambda}^{2k} \mathbf{V}^T$

B(4): $(\mathbf{A}^T \mathbf{A})^k \sim v_1 \lambda_1^{2k} v_1^T$

B(5): $(\mathbf{A}^T \mathbf{A})^k \mathbf{v'} \sim$ (constant) $\mathbf{v}_1$

CMU SCS

Least obvious properties

A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \mathbf{\Lambda}_{[r \times r]} \mathbf{V^T}_{[r \times m]}$

C(1): $\mathbf{A}_{[n \times m]} \mathbf{x}_{[m \times 1]} = \mathbf{b}_{[n \times 1]}$

let $\mathbf{x}_0 = \mathbf{V} \mathbf{\Lambda}^{(-1)} \mathbf{U}^T \mathbf{b}$

if under-specified, $\mathbf{x}_0$ gives 'shortest' solution

if over-specified, it gives the 'solution' with the smallest least squares error

(see Num. Recipes, p. 62)

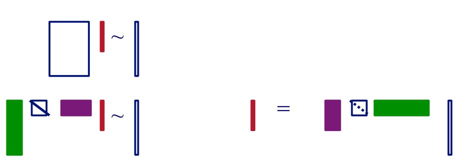
CMU SCS

Least obvious properties

A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \mathbf{\Lambda}_{[r \times r]} \mathbf{V^T}_{[r \times m]}$

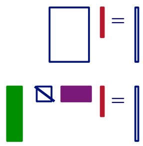
C(1): $\mathbf{A}_{[n \times m]} \mathbf{x}_{[m \times 1]} = \mathbf{b}_{[n \times 1]}$

let $\mathbf{x}_0 = \mathbf{V} \mathbf{\Lambda}^{(-1)} \mathbf{U}^T \mathbf{b}$



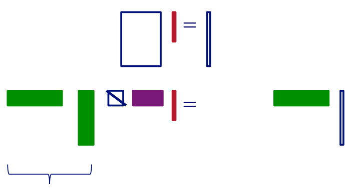
CMU SCS

Slowly:



CMU SCS

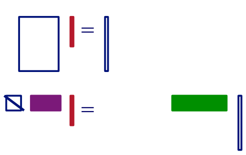
Slowly:



Identity
U: column-
orthonormal

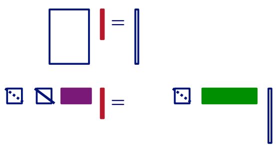
CMU SCS

Slowly:



CMU SCS

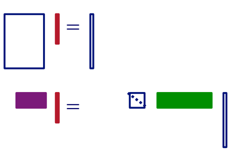
Slowly:



15-826 Copyright: C. Faloutsos (2012) 31

CMU SCS

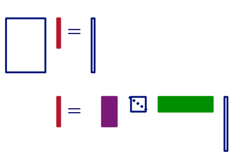
Slowly:



15-826 Copyright: C. Faloutsos (2012) 32

CMU SCS

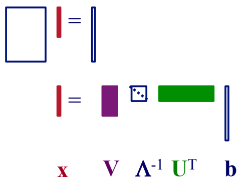
Slowly:



15-826 Copyright: C. Faloutsos (2012) 33

CMU SCS

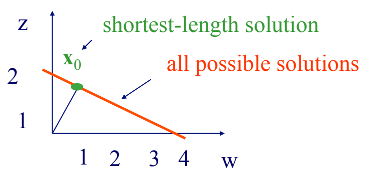
Slowly:



CMU SCS

Least obvious properties

Illustration: under-specified, eg
[1 2] [w z] ^{T} = 4 (ie, 1 w + 2 z = 4)



CMU SCS

Exercise

Verify formula:

A = [1 2] **b** = [4]
A = **U** **Λ** **V**^{T}
U = ??
Λ = ??
V= ??
$\mathbf{x_0}$ = **V** **Λ** $^{(-1)}$ **U**^{T} **b**

CMU SCS

Exercise

Verify formula:

$\mathbf{A}$ = [1 2] $\mathbf{b}$ = [4]

$\mathbf{A} = \mathbf{U}\,\mathbf{\Lambda}\,\mathbf{V}^T$

$\mathbf{U}$ = [1]

$\mathbf{\Lambda}$ = [sqrt(5)]

$\mathbf{V}$= [1/sqrt(5)   2/sqrt(5)] ^{T}

$\mathbf{x}_0 = \mathbf{V}\,\mathbf{\Lambda}^{(-1)}\,\mathbf{U}^T\,\mathbf{b}$

CMU SCS

Exercise

Verify formula:

$\mathbf{A}$ = [1 2] $\mathbf{b}$ = [4]

$\mathbf{A} = \mathbf{U}\,\mathbf{\Lambda}\,\mathbf{V}^T$

$\mathbf{U}$ = [1]

$\mathbf{\Lambda}$ = [sqrt(5)]

$\mathbf{V}$= [1/sqrt(5)   2/sqrt(5)] ^{T}

$\mathbf{x}_0 = \mathbf{V}\,\mathbf{\Lambda}^{(-1)}\,\mathbf{U}^T\,\mathbf{b}$ **=** [1/5   2/5] ^{T} [4]

= [4/5 8/5] ^{T}: w= 4/5, z = 8/5

CMU SCS

Exercise

Verify formula:

Show that w= 4/5, z = 8/5 is

(a) A solution to 1*w + 2*z = 4 and

(b) Minimal (wrt Euclidean norm)

CMU SCS

Exercise

Verify formula:

Show that w= 4/5, z = 8/5 is

(a) A solution to 1*w + 2*z = 4 and

A: easy

(b) Minimal (wrt Euclidean norm)

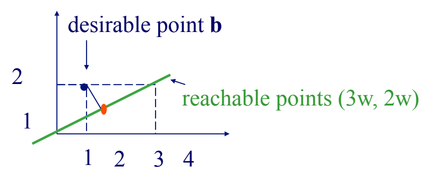
A: [4/5 8/5] is perpenticular to [2 -1]

CMU SCS

Least obvious properties – cont'd

Illustration: over-specified, eg

$[3\ 2]^T\ [w] = [1\ 2]^T$ (ie, 3 w = 1; 2 w = 2)



CMU SCS

Exercise

Verify formula:

$\mathbf{A} = [3\ 2]^T \quad \mathbf{b} = [1\ \ 2]^T$

$\mathbf{A} = \mathbf{U}\ \mathbf{\Lambda}\ \mathbf{V}^T$

$\mathbf{U}$ = ??

$\mathbf{\Lambda}$ = ??

$\mathbf{V}$ = ??

$\mathbf{x_0} = \mathbf{V}\ \mathbf{\Lambda}^{(-1)}\ \mathbf{U}^T\ \mathbf{b}$

CMU SCS

Exercise

Verify formula:

$\mathbf{A} = [3\ 2]^T \quad \mathbf{b} = [1\ \ 2]^T$

$\mathbf{A} = \mathbf{U}\, \mathbf{\Lambda}\, \mathbf{V}^T$

$\mathbf{U} = [\ 3/\mathrm{sqrt}(13) \quad 2/\mathrm{sqrt}(13)\ ]^T$

$\mathbf{\Lambda} = [\ \mathrm{sqrt}(13)\ ]$

$\mathbf{V} = [\ 1\ ]$

$\mathbf{x_0} = \mathbf{V}\, \mathbf{\Lambda}^{(-1)}\ \mathbf{U}^T\, \mathbf{b} = [\ 7/13\ ]$

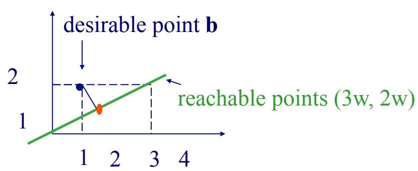
CMU SCS

Exercise

Verify formula:

$[3\ 2]^T\ \ [7/13] = [1\ 2]^T$

$[21/13\ \ 14/13\ ]^T$ -> 'red point'



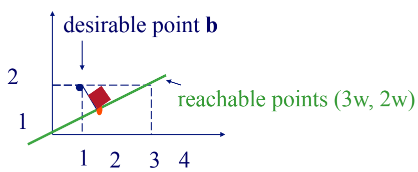
CMU SCS

Exercise

Verify formula:

$[3\ 2]^T\ \ [7/13] = [1\ 2]^T$

$[21/13\ \ 14/13\ ]^T$ -> 'red point' - <u>perpenticular</u>?

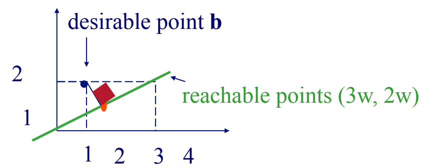


CMU SCS

Exercise

Verify formula:

A: [3 2] . ([1 2] – [21/13 14/13]) =
[3 2] . [-8/13 12/13] = [3 2] . [-2 3] = 0



15-826 Copyright: C. Faloutsos (2012) 46

CMU SCS

Least obvious properties - cont'd

A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \mathbf{\Lambda}_{[r \times r]} \mathbf{V}^T_{[r \times m]}$

C(2): $\mathbf{A}_{[n \times m]} \mathbf{v_1}_{[m \times 1]} = \lambda_1 \mathbf{u_1}_{[n \times 1]}$
where $\mathbf{v}_1$, $\mathbf{u}_1$ the first (column) vectors of **V**, **U**. ($\mathbf{v}_1$ == right-singular-vector)

C(3): symmetrically: $\mathbf{u}_1^T \mathbf{A} = \lambda_1 \mathbf{v}_1^T$
$\mathbf{u}_1$ == left-singular-vector

Therefore:

15-826 Copyright: C. Faloutsos (2012) 47

CMU SCS

Least obvious properties - cont'd

A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \mathbf{\Lambda}_{[r \times r]} \mathbf{V}^T_{[r \times m]}$

$\boxed{\text{C(4): } \mathbf{A}^T \mathbf{A}\, \mathbf{v}_1 = \lambda_1^2\, \mathbf{v}_1}$

(**fixed point** - the dfn of eigenvector for a symmetric matrix)

15-826 Copyright: C. Faloutsos (2012) 48

CMU SCS

Least obvious properties - altogether

A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \, \boldsymbol{\Lambda}_{[r \times r]} \, \mathbf{V}^{\mathbf{T}}_{[r \times m]}$

C(1): $\mathbf{A}_{[n \times m]} \, \mathbf{x}_{[m \times 1]} = \mathbf{b}_{[n \times 1]}$
then, $\mathbf{x}_0 = \mathbf{V} \, \boldsymbol{\Lambda}^{(-1)} \, \mathbf{U}^{T} \, \mathbf{b}$**:** shortest, actual or least-squares solution

C(2): $\mathbf{A}_{[n \times m]} \, \mathbf{v}_{\mathbf{1}\,[m \times 1]} = \boldsymbol{\lambda}_{\mathbf{1}} \, \mathbf{u}_{\mathbf{1}\,[n \times 1]}$

C(3): $\mathbf{u}_{\mathbf{1}}^{\mathbf{T}} \, \mathbf{A} = \boldsymbol{\lambda}_{\mathbf{1}} \, \mathbf{v}_{\mathbf{1}}^{\mathbf{T}}$

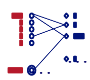
C(4): $\mathbf{A}^{\mathbf{T}} \, \mathbf{A} \, \mathbf{v}_1 = \boldsymbol{\lambda}_1^{\mathbf{2}} \, \mathbf{v}_1$

15-826 Copyright: C. Faloutsos (2012) 49

CMU SCS

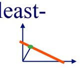
Properties - conclusions

A(0): $\mathbf{A}_{[n \times m]} = \mathbf{U}_{[n \times r]} \, \boldsymbol{\Lambda}_{[r \times r]} \, \mathbf{V}^{\mathbf{T}}_{[r \times m]}$



B(5): $(\mathbf{A}^{T} \, \mathbf{A})^{k} \, \mathbf{v'} \sim \text{(constant)} \, \mathbf{v}_1$

C(1): $\mathbf{A}_{[n \times m]} \, \mathbf{x}_{[m \times 1]} = \mathbf{b}_{[n \times 1]}$
then, $\mathbf{x}_0 = \mathbf{V} \, \boldsymbol{\Lambda}^{(-1)} \, \mathbf{U}^{T} \, \mathbf{b}$**:** shortest, actual or least-squares solution



C(4): $\mathbf{A}^{\mathbf{T}} \, \mathbf{A} \, \mathbf{v}_1 = \boldsymbol{\lambda}_1^{\mathbf{2}} \, \mathbf{v}_1$

15-826 Copyright: C. Faloutsos (2012) 50

CMU SCS

SVD - detailed outline

- ...
- Case studies
- SVD properties
- more case studies
  - Kleinberg/google algorithms
  - query feedbacks
- Conclusions

15-826 Copyright: C. Faloutsos (2012) 51

CMU SCS

Kleinberg's algo (HITS)

Kleinberg, Jon (1998). *Authoritative sources in a hyperlinked environment*. Proc. 9th ACM-SIAM Symposium on Discrete Algorithms.

15-826 Copyright: C. Faloutsos (2012) 52

CMU SCS

Kleinberg's algorithm

- Problem dfn: given the web and a query
- find the most 'authoritative' web pages for this query

Step 0: find all pages containing the query terms

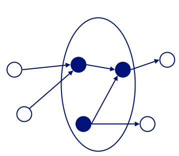
Step 1: expand by one move forward and backward

15-826 Copyright: C. Faloutsos (2012) 53

CMU SCS

Kleinberg's algorithm

- Step 1: expand by one move forward and backward

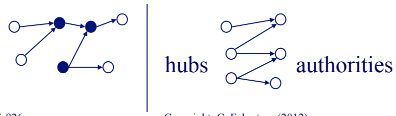


15-826 Copyright: C. Faloutsos (2012) 54

CMU SCS

Kleinberg's algorithm

- on the resulting graph, give high score (= 'authorities') to nodes that many important nodes point to
- give high importance score ('hubs') to nodes that point to good 'authorities')



CMU SCS

Kleinberg's algorithm

observations

- recursive definition!
- each node (say, 'i'-th node) has both an authoritativeness score a_i and a hubness score h_i

CMU SCS

Kleinberg's algorithm

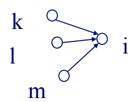
Let E be the set of edges and $\mathbf{A}$ be the adjacency matrix:

the (i,j) is 1 if the edge from i to j exists

Let h and a be [n x 1] vectors with the 'hubness' and 'authoritativiness' scores.

Then:

Kleinberg’s algorithm



Then:

$$a_i = h_k + h_l + h_m$$

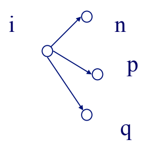
that is

a_i = Sum (h_j) over all j that (j,i) edge exists

or

$\mathbf{a} = \mathbf{A}^T \mathbf{h}$

Kleinberg’s algorithm



symmetrically, for the ‘hubness’:

$$h_i = a_n + a_p + a_q$$

that is

h_i = Sum (q_j) over all j that (i,j) edge exists

or

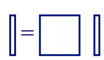
$\mathbf{h} = \mathbf{A}\ \mathbf{a}$

Kleinberg’s algorithm

In conclusion, we want vectors **h** and **a** such that:

$$\mathbf{h} = \mathbf{A}\ \mathbf{a}$$

$$\mathbf{a} = \mathbf{A}^T \mathbf{h}$$



Recall properties:

C(2): $\mathbf{A}_{[n \times m]}\ \mathbf{v_1}_{[m \times 1]} = \lambda_1\ \mathbf{u_1}_{[n \times 1]}$

C(3): $\mathbf{u_1}^T \mathbf{A} = \lambda_1\ \mathbf{v_1}^T$

CMU SCS

Kleinberg's algorithm

In short, the solutions to

$$\mathbf{h} = \mathbf{A}\ \mathbf{a}$$

$$\mathbf{a} = \mathbf{A}^T\ \mathbf{h}$$

are the left- and right- singular-vectors of the adjacency matrix **A.**

Starting from random **a'** and iterating, we'll eventually converge

(Q: to which of all the singular-vectors? why?)

CMU SCS

Kleinberg's algorithm

(Q: to which of all the singular-vectors? why?)

A: to the ones of the strongest singular-value, because of property B(5):

B(5): $(\mathbf{A}^T \mathbf{A})^k\ \mathbf{v'} \sim \text{(constant)}\ \mathbf{v}_1$

CMU SCS

Kleinberg's algorithm - results

Eg., for the query 'java':

0.328 www.gamelan.com

0.251 java.sun.com

0.190 www.digitalfocus.com ("the java developer")

CMU SCS

Kleinberg's algorithm - discussion

- 'authority' score can be used to find 'similar pages' (how?)
- closely related to 'citation analysis', social networs / 'small world' phenomena

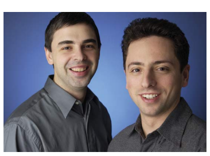
CMU SCS

SVD - detailed outline

- ...
- Case studies
- SVD properties
- more case studies
  - Kleinberg/google algorithms
  - query feedbacks
- Conclusions

CMU SCS

PageRank (google)



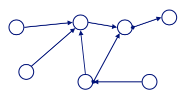
Larry Page Sergey Brin

•Brin, Sergey and Lawrence Page (1998). *Anatomy of a Large-Scale Hypertextual Web Search Engine*. 7th Intl World Wide Web Conf.

CMU SCS

Problem: PageRank

Given a directed graph, find its most interesting/central node



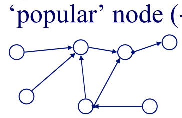
A node is important, if it is connected with important nodes (recursive, but OK!)

CMU SCS

Problem: PageRank - solution

Given a directed graph, find its most interesting/central node

Proposed solution: Random walk; spot most 'popular' node (-> steady state prob. (ssp))

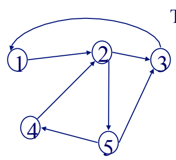


A node has high ssp, if it is connected with high ssp nodes (recursive, but OK!)

CMU SCS

(Simplified) PageRank algorithm

- Let **A** be the adjacency matrix;
- let **B** be the transition matrix: transpose, column-normalized - then



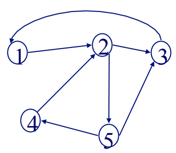
From / To, **B**

$$\begin{bmatrix} & & 1 & & \\ 1 & & & 1 & \\ & 1/2 & & & 1/2 \\ & & & & 1/2 \\ & 1/2 & & & \end{bmatrix} \begin{bmatrix} p1 \\ p2 \\ p3 \\ p4 \\ p5 \end{bmatrix} = \begin{bmatrix} p1 \\ p2 \\ p3 \\ p4 \\ p5 \end{bmatrix}$$

CMU SCS

(Simplified) PageRank algorithm

- **B p = p**



$$\underbrace{\begin{bmatrix} & & 1 & & \\ 1 & & & 1 & \\ & 1/2 & & & 1/2 \\ & & & & 1/2 \\ & 1/2 & & & \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} p1 \\ p2 \\ p3 \\ p4 \\ p5 \end{bmatrix}}_{\mathbf{p}} = \underbrace{\begin{bmatrix} p1 \\ p2 \\ p3 \\ p4 \\ p5 \end{bmatrix}}_{\mathbf{p}}$$

CMU SCS

(Simplified) PageRank algorithm

- **B p** = 1 * **p**
- thus, **p** is the **eigenvector** that corresponds to the highest eigenvalue (=1, since the matrix is column-normalized)
- Why does such a **p** exist?
  - **p** exists if **B** is nxn, nonnegative, irreducible [Perron–Frobenius theorem]

CMU SCS

(Simplified) PageRank algorithm

- **B p** = 1 * **p**
- thus, **p** is the **eigenvector** that corresponds to the highest eigenvalue (=1, since the matrix is column-normalized)
- Why does such a **p** exist?
  - **p** exists if **B** is nxn, nonnegative, irreducible [Perron–Frobenius theorem]

All<-> all

CMU SCS

(Simplified) PageRank algorithm

- In short: imagine a particle randomly moving along the edges
- compute its steady-state probabilities (ssp)

Full version of algo: with occasional random jumps

Why? To make the matrix irreducible

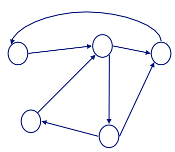
CMU SCS

Full Algorithm

- With probability *1-c*, fly-out to a random node
- Then, we have

$\mathbf{p} = c\,\mathbf{B}\,\mathbf{p} + (1\text{-}c)/n\ \mathbf{1} \Rightarrow$

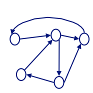
$\mathbf{p} = (1\text{-}c)/n\ [\mathbf{I} - c\,\mathbf{B}]^{-1}\ \mathbf{1}$



CMU SCS

Full Algorithm

- With probability *1-c*, fly-out to a random node
- Then, we have



$\mathbf{p} = c\,\mathbf{B}\,\mathbf{p} + (1\text{-}c)/n\ \mathbf{1} \Rightarrow$

$\mathbf{p} = (1\text{-}c)/n\ [\mathbf{I} - c\,\mathbf{B}]^{-1}\ \mathbf{1}$

$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

CMU SCS

Alternative notation – eigenvector viewpoint

M Modified transition matrix

$\mathbf{M} = c\,\mathbf{B} + (1\text{-}c)/n\ \mathbf{1}\ \mathbf{1}^T$

CLIQUE

Then

$\mathbf{p} = \mathbf{M}\,\mathbf{p}$

That is: the steady state probabilities =

PageRank scores form the *first eigenvector* of the ‘modified transition matrix’

15-826 Copyright: C. Faloutsos (2012) 76

CMU SCS

Parenthesis: intuition behind eigenvectors

- Definition
- 3 properties
- intuition

15-826 Copyright: C. Faloutsos (2012) 77

CMU SCS

Formal definition

If **A** is a (n x n) square matrix

(λ , **x**) is an **eigenvalue/eigenvector** pair of **A** if

$$\boxed{\mathbf{A}\,\mathbf{x} = \lambda\,\mathbf{x}}$$

CLOSELY related to singular values:

15-826 Copyright: C. Faloutsos (2012) 78

CMU SCS

Property #1: Eigen- vs singular-values

if

$$\mathbf{B}_{[n \times m]} = \mathbf{U}_{[n \times r]} \, \mathbf{\Lambda}_{[r \times r]} \, (\mathbf{V}_{[m \times r]})^T$$

then $\mathbf{A} = (\mathbf{B}^T\mathbf{B})$ is symmetric and

$$C(4): \mathbf{B}^T \mathbf{B} \, \mathbf{v}_i = \lambda_i^2 \, \mathbf{v}_i$$

ie, $\mathbf{v}_1$, $\mathbf{v}_2$, ...: eigenvectors of $\mathbf{A} = (\mathbf{B}^T\mathbf{B})$

15-826 Copyright: C. Faloutsos (2012) 79

CMU SCS

Property #2

- If $\mathbf{A}_{[nxn]}$ is a real, symmetric matrix
- Then it has *n* real eigenvalues

(if **A** is not symmetric, some eigenvalues may be complex)

15-826 Copyright: C. Faloutsos (2012) 80

CMU SCS

Property #3

- If $\mathbf{A}_{[nxn]}$ is a real, symmetric matrix
- Then it has *n* real eigenvalues
- And they agree with its *n* singular values, except possibly for the sign

15-826 Copyright: C. Faloutsos (2012) 81

CMU SCS

Parenthesis: intuition behind eigenvectors

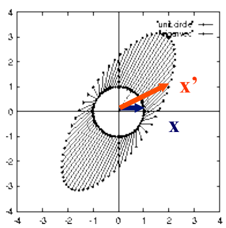
- Definition
- 3 properties
- **intuition**

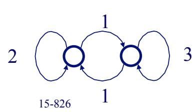
CMU SCS

Intuition

- **A** as vector transformation

$$\begin{array}{ccccc} \mathbf{x'} & & \mathbf{A} & & \mathbf{x} \\ \begin{bmatrix} 2 \\ 1 \end{bmatrix} & = & \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} & & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array}$$



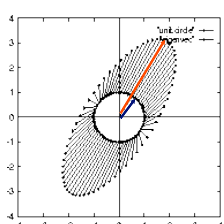


CMU SCS

Intuition

- By defn., eigenvectors remain parallel to themselves ('**fixed points**')

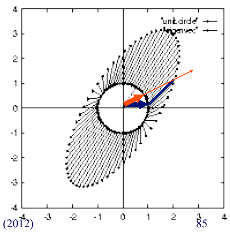
$$\begin{array}{ccccccc} \lambda_1 & & \mathbf{v_1} & & \mathbf{A} & & \mathbf{v_1} \\ 3.62 & * & \begin{bmatrix} 0.52 \\ 0.85 \end{bmatrix} & = & \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} & & \begin{bmatrix} 0.52 \\ 0.85 \end{bmatrix} \end{array}$$



CMU SCS

Convergence

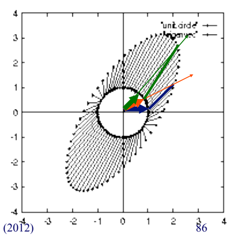
- Usually, fast:



CMU SCS

Convergence

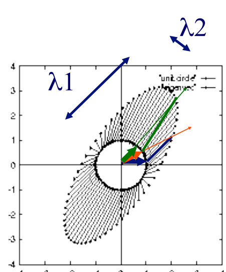
- Usually, fast:



CMU SCS

Convergence

- Usually, fast:
- depends on ratio
  $\lambda 1 : \lambda 2$



CMU SCS

Closing the parenthesis wrt intuition behind eigenvectors

15-826 Copyright: C. Faloutsos (2012) 88

CMU SCS

Kleinberg/PageRank - conclusions

SVD helps in graph analysis:

hub/authority scores: strongest left- and right-singular-vectors of the adjacency matrix

random walk on a graph: steady state probabilities are given by the strongest eigenvector of the transition matrix

15-826 Copyright: C. Faloutsos (2012) 89

CMU SCS

SVD - detailed outline

- ...
- Case studies
- SVD properties
- more case studies
  - google/Kleinberg algorithms
  - query feedbacks
- Conclusions

15-826 Copyright: C. Faloutsos (2012) 90

CMU SCS

Query feedbacks

[Chen & Roussopoulos, sigmod 94]

Sample problem:

estimate selectivities (e.g., '*how many movies were made between 1940 and 1945?*'

for query optimization,

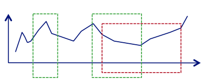
LEARNING from the query results so far!!

15-826 Copyright: C. Faloutsos (2012) 91

CMU SCS

Query feedbacks

- Given: past queries and their results
  - #movies(1925,1935) = 52
  - #movies(1948, 1990) = 123
  - …
  - And a new query, say #movies(1979,1980)?
- Give your best estimate

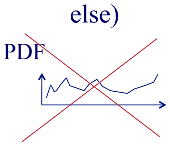


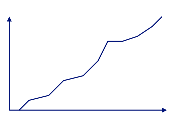
15-826 Copyright: C. Faloutsos (2012) 92

CMU SCS

Query feedbacks

Idea #1: consider a function for the CDF (cummulative distr. function), eg., 6-th degree polynomial (or splines, or anything else)





15-826 Copyright: C. Faloutsos (2012) 93

CMU SCS

Query feedbacks

For example

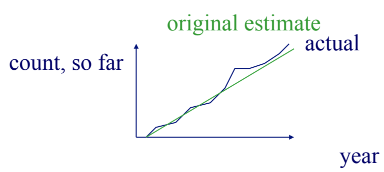
F(x) = # movies made until year 'x'

$= a_1 + a_2 * x + a_3 * x^2 + \ldots a_7 * x^6$

CMU SCS

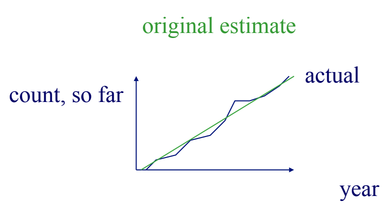
Query feedbacks

GREAT idea #2: adapt your model, as you see the actual counts of the actual queries



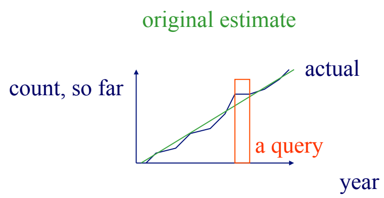
CMU SCS

Query feedbacks



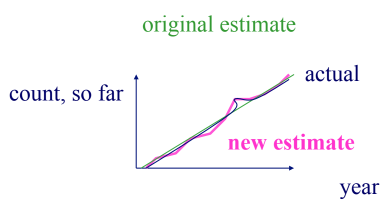
CMU SCS

Query feedbacks



CMU SCS

Query feedbacks



CMU SCS

Query feedbacks

Eventually, the problem becomes:

- estimate the parameters a_1, ... a_7 of the model
- to minimize the least squares errors from the real answers so far.

Formally:

CMU SCS

Query feedbacks

Formally, with *n* queries and 6-th degree polynomials:

$$\begin{bmatrix} X11 & X12 & & & X17 \\ & & & & \\ & & & & \\ Xn1 & Xn2 & & & Xn7 \end{bmatrix} \begin{bmatrix} a1 \\ a2 \\ \\ \\ a7 \end{bmatrix} = \begin{bmatrix} b1 \\ b2 \\ \\ \\ bn \end{bmatrix}$$

CMU SCS

Query feedbacks

where $x_{i,j}$ such that Sum ($x_{i,j}$ * a_i) = our estimate for the # of movies and b_j: the actual

$$\begin{bmatrix} X11 & X12 & & & X17 \\ & & & & \\ & & & & \\ Xn1 & Xn2 & & & Xn7 \end{bmatrix} \begin{bmatrix} a1 \\ a2 \\ \\ \\ a7 \end{bmatrix} = \begin{bmatrix} b1 \\ b2 \\ \\ \\ bn \end{bmatrix}$$

CMU SCS

Query feedbacks

For example, for query '*find the count of movies during (1920-1932)*':

$a_1 + a_2 * 1932 + a_3 * 1932**2 + \ldots$

-

$(a_1 + a_2 * 1920 + a_3 * 1920**2 + \ldots)$

$$\begin{bmatrix} X11 & X12 & & & X17 \\ & & & & \\ & & & & \\ Xn1 & Xn2 & & & Xn7 \end{bmatrix} \begin{bmatrix} a1 \\ a2 \\ \\ \\ a7 \end{bmatrix} = \begin{bmatrix} b1 \\ b2 \\ \\ \\ bn \end{bmatrix}$$

CMU SCS

Query feedbacks

And thus X11 = 0; X12 = 1932-1920, etc

$a_1 + a_2 * 1932 + a_3 * 1932**2 + \ldots$
-
$(a_1 + a_2 * 1920 + a_3 * 1920**2 + \ldots)$

$$\begin{bmatrix} X11 & X12 & & & X17 \\ & & & & \\ & & & & \\ & & & & \\ Xn1 & Xn2 & & & Xn7 \end{bmatrix} \begin{bmatrix} a1 \\ a2 \\ \\ \\ a7 \end{bmatrix} = \begin{bmatrix} b1 \\ b2 \\ \\ \\ bn \end{bmatrix}$$

CMU SCS

Query feedbacks

In matrix form:

X **a** = **b**

$$\begin{array}{l} \text{1st query} \\ \\ \\ \\ \text{n-th query} \end{array} \begin{bmatrix} X11 & X12 & & & X17 \\ & & & & \\ & & & & \\ & & & & \\ Xn1 & Xn2 & & & Xn7 \end{bmatrix} \begin{bmatrix} a1 \\ a2 \\ \\ \\ a7 \end{bmatrix} = \begin{bmatrix} b1 \\ b2 \\ \\ \\ bn \end{bmatrix}$$

CMU SCS

Query feedbacks

In matrix form:

$$\mathbf{X}\,\mathbf{a} = \mathbf{b}$$

and the least-squares estimate for **a** is

$$\mathbf{a} = \mathbf{V}\,\mathbf{\Lambda}^{(-1)}\,\mathbf{U}^{\mathbf{T}}\,\mathbf{b}$$

according to property C(1)

(let $\mathbf{X} = \mathbf{U}\,\mathbf{\Lambda}\,\mathbf{V}^{\mathbf{T}}$)

CMU SCS

Query feedbacks - enhancements

The solution

$$\mathbf{a} = \mathbf{V}\ \boldsymbol{\Lambda}^{(-1)}\ \mathbf{U}^{\mathbf{T}}\ \mathbf{b}$$

works, but needs expensive SVD each time a new query arrives

GREAT Idea #3: Use 'Recursive Least Squares', to adapt **a** incrementally.

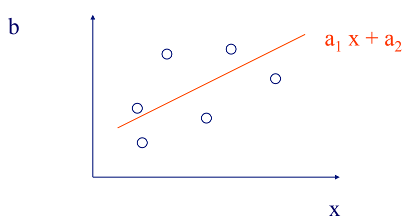
Details: in paper - intuition:

15-826 Copyright: C. Faloutsos (2012) 106

CMU SCS

Query feedbacks - enhancements

Intuition: least squares fit

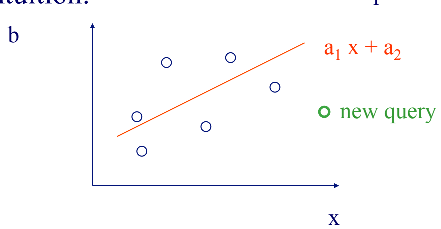


15-826 Copyright: C. Faloutsos (2012) 107

CMU SCS

Query feedbacks - enhancements

Intuition: least squares fit

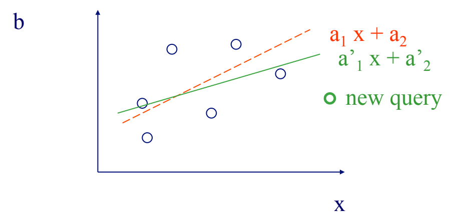


15-826 Copyright: C. Faloutsos (2012) 108

CMU SCS

Query feedbacks - enhancements

Intuition: least squares fit



CMU SCS

Query feedbacks - enhancements

the new coefficients can be quickly computed from the old ones, plus statistics in a (7x7) matrix

(no need to know the details, although the RLS is a brilliant method)

CMU SCS

Query feedbacks - enhancements

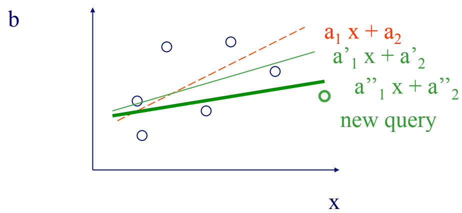
GREAT idea #4: ‘forgetting’ factor - we can even down-play the weight of older queries, since the data distribution might have changed.

(comes for ‘free’ with RLS...)

CMU SCS

Query feedbacks - enhancements

Intuition: least squares fit



CMU SCS

Query feedbacks - conclusions

SVD helps find the Least Squares solution, to adapt to query feedbacks

(RLS = Recursive Least Squares is a great method to incrementally update least-squares fits)

CMU SCS

SVD - detailed outline

- ...
- Case studies
- SVD properties
- more case studies
  - google/Kleinberg algorithms
  - query feedbacks
- Conclusions

CMU SCS

Conclusions

- SVD: a **valuable** tool
- given a document-term matrix, it finds 'concepts' (LSI)
- ... and can reduce dimensionality (KL)
- ... and can find rules (PCA; RatioRules)

15-826 Copyright: C. Faloutsos (2012) 115

CMU SCS

Conclusions cont'd

- ... and can find fixed-points or steady-state probabilities (google/ Kleinberg/ Markov Chains)
- ... and can solve optimally over- and under-constraint linear systems (least squares / query feedbacks)

15-826 Copyright: C. Faloutsos (2012) 116

CMU SCS

References

- Brin, S. and L. Page (1998). Anatomy of a Large-Scale Hypertextual Web Search Engine. 7th Intl World Wide Web Conf.
- Chen, C. M. and N. Roussopoulos (May 1994). Adaptive Selectivity Estimation Using Query Feedback. Proc. of the ACM-SIGMOD , Minneapolis, MN.

15-826 Copyright: C. Faloutsos (2012) 117

References cont'd

- Kleinberg, J. (1998). Authoritative sources in a hyperlinked environment. Proc. 9th ACM-SIAM Symposium on Discrete Algorithms.
- Press, W. H., S. A. Teukolsky, et al. (1992). Numerical Recipes in C, Cambridge University Press.

