15-859(B) Machine Learning Theory

Homework # 3 **Due: February 18, 2008**

Groundrules: Same as before. You should work on the exercises by yourself but may work with others on the problems (just write down who you worked with). Also if you use material from outside sources, say where you got it.

Exercises:

1. What is the VC-dimension d of axis-parallel rectangles in R^3? Specifically, a legal target function is specified by three intervals $[x_{min}, x_{max}]$, $[y_{min}, y_{max}]$, and $[z_{min}, z_{max}]$, and classifies an example (x, y, z) as positive iff $x \in [x_{min}, x_{max}]$, $y \in [y_{min}, y_{max}]$, and $z \in [z_{min}, z_{max}]$. Be sure to argue why no set of $d + 1$ points can be shattered.

   (It may help to first think about the case of R^2, which is also in the book.)

Problems: In the problems below you will prove that the VC-dimension of the class H_n of halfspaces in n dimensions is $n+1$. (H_n is the set of functions $a_1x_1 + \ldots + a_nx_n \geq a_0$, where $a_0, \ldots, a_n$ are real-valued.) We will use the following definition: The *convex hull* of a set of points S is the set of all convex combinations of points in S; this is the set of all points that can be written as $\sum_{x_i \in S} \lambda_i x_i$, where each $\lambda_i \geq 0$, and $\sum_i \lambda_i = 1$. It is not hard to see that if a halfspace has all points from a set S on one side, then the entire convex hull of S must be on that side as well.

2. [**lower bound**] Prove that VC-dim$(H_n) \geq n+1$ by presenting a set of $n+1$ points in n-dimensional space such that one can partition that set with halfspaces in all possible ways. (And, show how one can partition the set in any desired way.)

3. [**upper bound part 1**] The following is "Radon's Theorem," from the 1920's.

   Theorem. *Let S be a set of $n+2$ points in n dimensions. Then S can be partitioned into two (disjoint) subsets S_1 and S_2 whose convex hulls intersect.*

   Show that Radon's Theorem implies that the VC-dimension of halfspaces is *at most* $n + 1$. Conclude that VC-dim$(H_n) = n + 1$.

4. [**upper bound part 2**] Now we prove Radon's Theorem. We will need the following standard fact from linear algebra. If $x_1, \ldots, x_{n+1}$ are $n + 1$ points in n-dimensional space, then they are linearly dependent. That is, there exist real values $\lambda_1, \ldots, \lambda_{n+1}$ *not all zero* such that $\lambda_1 x_1 + \ldots + \lambda_{n+1} x_{n+1} = 0$.

   You may now prove Radon's Theorem however you wish. However, as a suggested first step, prove the following. For any set of $n + 2$ points $x_1, \ldots, x_{n+2}$ in n-dimensional space, there exist $\lambda_1, \ldots, \lambda_{n+2}$ *not all zero* such that $\sum_i \lambda_i x_i = 0$ and $\sum_i \lambda_i = 0$. (This is called *affine dependence*.) Now, think about the lambdas...