$\Omega(\frac{\log n}{\log \log n})$ Integrality Gap for Priority Steiner

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1 The Gap Example

There are k priorities, with level 1 being highest and level k lowest. There are $\approx N = (2k)^k$ nodes (which are not terminals) and we define a set of $\approx N$ terminals as well. The nodes will be defined by repeatedly subdividing a line of unit length; this line has edges of level 1.

- **Level 1.** Divide the line into $2k$ parts of length $1/2k$ each: hence there are $2k+1$ equi-spaced nodes on the line. Create one terminal at level 1. Attach the odd nodes to this terminal and the even nodes to the root, using edges of level 1 and cost 0. Note that this creates k disjoint paths from the terminal to the root, and you will send $1/k$ flow on each such path.
- **Level 2.** For each segment, subdivide it into $2k$ parts (now of length $1/(2k)^2$ each) by adding $2k-1$ nodes. Create a new level-2 terminal for each segment and attach this terminal to the odd-numbered newly created nodes, and attach the even numbered new nodes (and the two older nodes at the endpoints of the segment) to the root: all the new edges are level 2 edges with cost 0. Again, note that each new level-2 terminal has k disjoint paths to the root.
- **Level i.** Each segment of length $1/(2k)^{i-1}$ is broken into subsegments of length $1/(2k)^i$, etc.

The fractional solution buys $1/k$ of each edge. The only non-zero cost edges are those on the line, and since they are used to only $1/k$, the fractional cost is $1/k$.

In the integral solution, one level-1 path is bought (of length $1/2k$). On each of the $(2k-1)$ segments of length $1/2k$, we have to buy another path of length $1/(2k)^2$. On the remaining $(2k)^2-(2k-1)-2k = (2k-1)^2$ segments of length, we buy paths of length $1/(2k)^3$, etc. The total cost is

$$\tfrac{1}{2k} + \textstyle\sum_{i=2}^{k} (2k-1)^{i-1} \times \tfrac{1}{(2k)^i} \geq \tfrac{1}{2k} + \textstyle\sum_{i=2}^{k} (1-\tfrac{1}{2k})^{i-1} \times \tfrac{1}{2k} \approx k \cdot \Theta(1/k) = \Theta(1). \tag{1.1}$$

This proves the following result:

Theorem 1.1 *The natural LP relaxation for the priority Steiner problem has an integrality gap of* $\Omega(\frac{\log n}{\log \log n})$.