More on Lattices

Reading: NNH 2.3

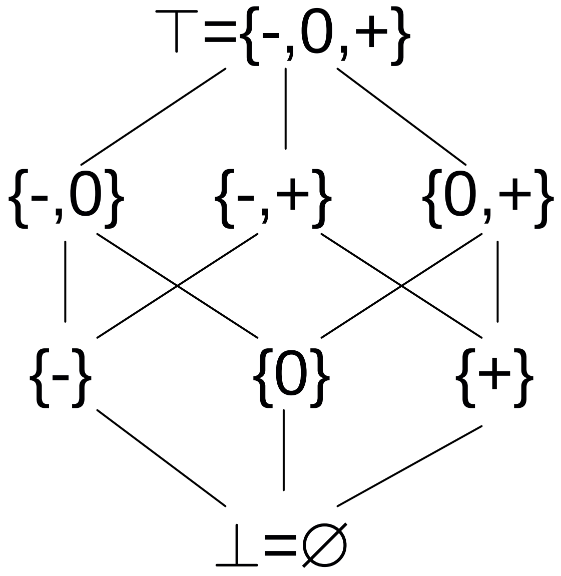
17-654/17-765
Analysis of Software Artifacts
Jonathan Aldrich

Announcements

- Homework due at midnight
  - Any questions?

Sign Analysis

Subset Lattice



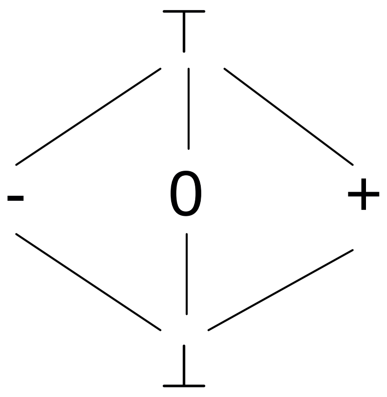
⊤={-,0,+}

{-,0} {-,+} {0,+}

{-} {0} {+}

⊥=∅

Custom Lattice



⊤

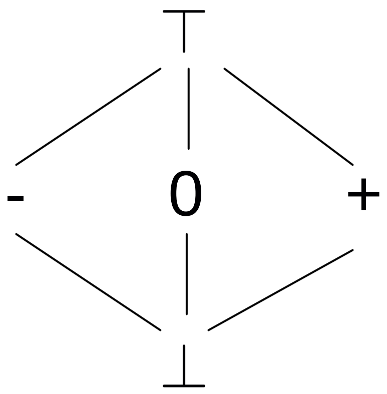
- 0 +

⊥

- Custom lattice exchanges precision for performance
  - Merges some subset lattice values into ⊤
  - Reduces lattice height => faster fixed point, less storage

Formal Definition

Custom Lattice



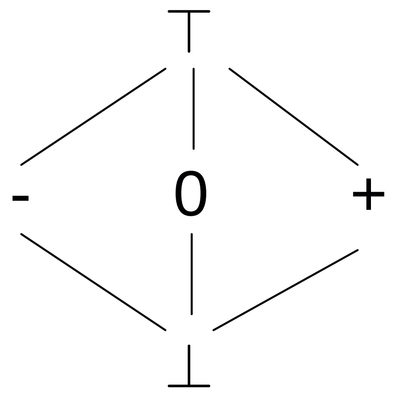
- Lattice is a tuple of custom lattices
  - One for each variable in the program
- Forward analysis
- Injected tuple ι = <⊤, …, ⊤>
  - Assume the worst of input variables
  - Could also assume ι = <0, …, 0> to model Java initialization
- Simple transfer functions *(σ is input data flow value)*
  - Can't use kill and gen sets, because *data flow values aren't sets!*

  - $f^{SA}([x := a],\sigma) = \sigma\ [x \mapsto SA(a,\sigma)]$
  - $f^{SA}([skip],\sigma) = \sigma$
  - $f^{SA}([b],\sigma) = \sigma$ *// could do better with separate T/F functions*

  - SA(n,σ) = *sign*(n) *// returns sign of n*
  - SA(x,σ) = σ(x)
  - SA(a_1 op_a $a_{2,}$$\sigma$) = ⊤ *// could do better by modeling each op*

Discussion

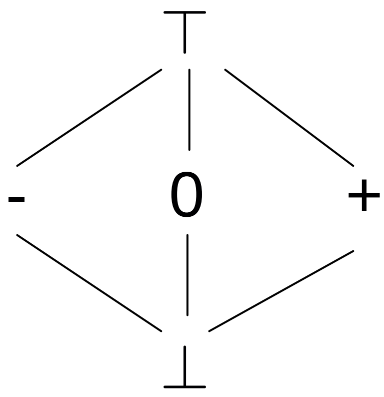
Custom Lattice



- Monotonicity
  - When a data flow value at a program point is recomputed, the new value is always ⊒ the old one
- ⊥
  - Most precise value possible
  - Always initial value at each program point
  - Typically means "haven't looked at this program point yet" (sometimes has domain semantics)
- ⊤
  - Least precise value (but always safe)
  - Typically means "don't know anything, and will never know more for this program point"
- +
  - Means "I believe this is +, but that might change to ⊤ as I iterate over different program paths"
  - Similar for -, 0
- Injected value ι
  - Depends on assumptions of analysis
    - sometimes ⊤, sometimes ⊥, sometimes another value

What if Initial Value were $\top$?

Custom Lattice



$[x := 5]^1$

while $[x < 10]^2$ do

$\quad [x := x+1]^3$

$[y := x]^4$

Initialize everything to $<(x,\top), (y,\top)>$

$SA_{enter}[1] = \iota = <(x,\top), (y,\top)>$

$SA_{exit}[1] = <(x,+), (y,\top)>$

$SA_{enter}[2] = SA_{exit}[1] \sqcup SA_{exit}[3]$

$\quad = <(x,+), (y,\top)> \sqcup <(x,\top), (y,\top)>$

$\quad = <(x,+ \sqcup \top), (y,\top \sqcup \top)>$

$\quad = <(x,\top), (y,\top)>$

…

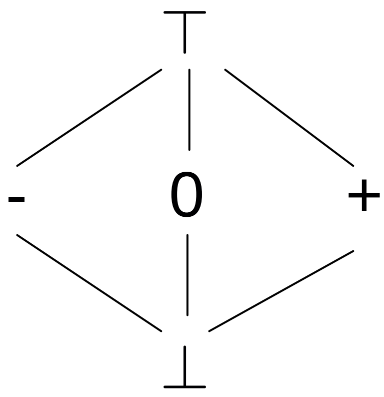
Everything else evaluates to $<(x,\top), (y,\top)>$

This fixed point is safe,

but we know that $x > 0$!

Initial Value Should Be ⊥

Custom Lattice



```
$[x := 5]^1$
while $[x < 10]^2$ do
        $[x := x+1]^3$
$[y := x]^4$
```

Initialize everything to <(x,⊥), (y,⊥)>

$SA_{enter}[1] = \iota = \langle(x,\top), (y,\top)\rangle$

$SA_{exit}[1] = \langle(x,+), (y,\top)\rangle$

$SA_{enter}[2] = SA_{exit}[1] \sqcup SA_{exit}[3]$

$= \langle(x,+), (y,\top)\rangle \sqcup \langle(x,\bot), (y,\bot)\rangle$

$= \langle(x,+ \sqcup \bot), (y,\top \sqcup \bot)\rangle$

$= \langle(x,+), (y,\top)\rangle$

…

Everything else evaluates to <(x,+), (y,⊤)>

Better result (and still safe!)

Analysis Correctness

(continued)

Reading: NNH 2.2 (optional)

17-654/17-765

Analysis of Software Artifacts

Jonathan Aldrich

What does Correctness Mean?

- Intuition
  - At a fixed point, analysis results are a *conservative abstraction* of *program execution*
- *Soundness* condition
  - When data flow analysis reaches a fixed point F, then for all traces T and all times t in each trace, $\alpha(T,t) \sqsubseteq F(pp(T,t))$

Proving Soundness

- Thus far:
  - Formally define analysis
    - Lattice framework
  - Define abstraction function
    - Maps (trace,time) to a data flow lattice value
  - Formalize execution
    - Structured operational semantics
    - Execution traces
- Prove *local soundness* for flow functions
- Apply *global soundness theorem*
- Examples

Abstraction

- Abstraction function α
  - maps traces to data flow values at a certain time *t* in the trace
- $\alpha_{CP}(<p_1,m_1>\ldots<p_n,m_n>,t)$
  $= m_t$

- Also define program point function *pp*
- $pp(<p_1,m_1>\ldots<p_n,m_n>,t)$
  $= p_t$

t	pp	x	y	z
0	1	3	0	0
1	2	3	3	0
2	3	3	3	1
3	4	3	3	1
4	5	3	3	3
5	3	3	2	3
6	4	3	2	3
7	5	3	2	6
8	3	3	1	6
9	6	3	1	6
10	-	3	0	6

$\alpha_{CP}(T,0) = (x=3,y=0,z=0)$
$\alpha_{CP}(T,10) = (x=3,y=0,z=6)$

Semantics of Assignment in WHILE

$$\frac{}{([x := a]^{\ell}, \sigma) \to ([], \sigma[x \mapsto \mathcal{A}(a, \sigma)])}\ [ass]$$

- Start with a program [x := a]$^{\ell}$ and a store σ
  - Goal: rewrite to a new program and new store
- We execute [x := a]$^{\ell}$ resulting in:
  - The empty program []
  - Evaluate a with store σ to get $\mathcal{A}$(a, σ)
  - Update x's value to be $\mathcal{A}$(a, σ)
- Example: a = x+3, σ = (x=5,y=7)
  - We get the pair ([], (x=8,y=7))

WHILE Traces, Formally

- A trace for program S_1 and initial state σ_1 is either:
  - a finite sequence $(S_1, \sigma_1), \ldots, ([], \sigma_n)$, where $(S_i, \sigma_i) \rightarrow (S_{i+1}, \sigma_{i+1})$ for $i \in 1, \ldots, n\text{-}1$
  - an infinite sequence $(S_1, \sigma_1), \ldots, (S_i, \sigma_i), \ldots$ where $(S_i, \sigma_i) \rightarrow (S_{i+1}, \sigma_{i+1})$ for $i \geq 1$

- Slight notational simplification
  - We will abbreviate $(S_1, \sigma_1), \ldots, (S_n, \sigma_n)$ as $(\mathit{first}(S_1), \sigma_1), \ldots, (\mathit{first}(S_n), \sigma_n)$
    - Uses program counter labels instead of complete programs

Local Soundness

$$\begin{tikzcd}
d_{in} \arrow[r, "f_{DF}(\mathit{first}(S_i))"] & d_{out} \sqsupseteq \alpha_{DF}(T,i+1) \\
(S_i, \sigma_i) \arrow[u, dashed, "\alpha_{DF}"'] \arrow[r, "\rightarrow"] & (S_{i+1}, \sigma_{i+1}) \arrow[u, dashed, "\alpha_{DF}"']
\end{tikzcd}$$

To show:

if $(S_i, \sigma_i) \rightarrow (S_{i+1}, \sigma_{i+1}) \in T$

and $d_{in} = \alpha_{DF}(T,i)$

and $d_{out} = f_{DF}(\mathit{first}(S_i), d_{in})$

then $\alpha_{DF}(T,i+1) \sqsubseteq d_{out}$

Intuitively, translating from concrete to abstract and applying the flow function will safely approximate ($\sqsupseteq$) taking a step in the trace and translating from concrete to abstract

Local Soundness for Constant Propagation

To show:

- if $(S_i, \sigma_i) \rightarrow (S_{i+1}, \sigma_{i+1}) \in T$
- and $d_{in} = \alpha_{CP}(T,i)$
- and $d_{out} = f_{CP}(\mathit{first}(S_i), d_{in})$
- then $\alpha_{CP}(T,i+1) \sqsubseteq d_{out}$

- Case: $S_i = [x := a]^{\ell}$
  - $\sigma_{i+1} = \sigma_i\,[x \mapsto \mathcal{A}(a, \sigma_i)]$
  - $d_{in} = \alpha_{CP}(T,i) = \sigma_i$
  - $d_{out} = f_{CP}([x := a]^{\ell}, \sigma_i)$
    $= \sigma_i\,[x \mapsto CP(a, \sigma_i)]$
  - $\alpha_{CP}(T,i+1) = \sigma_{i+1}$
    $= \sigma_i\,[x \mapsto \mathcal{A}(a, \sigma_i)]$
  - Lemma: $\mathcal{A}(a, \sigma_i) = CP(a, \sigma_i)$
  - Thus $\sigma_i\,[x \mapsto \mathcal{A}(a, \sigma_i)]$
    $\sqsubseteq \sigma_i\,[x \mapsto CP(a, \sigma_i)]$

What does Correctness Mean?

- Intuition
  - At a fixed point, analysis results are a *conservative abstraction* of *program execution*
- *Soundness* condition
  - When data flow analysis reaches a fixed point *F*, then for all traces *T* and all times *t* in each trace, $\alpha(T,t) \sqsubseteq F(pp(T,t))$

Global Soundness

- Intuition
  - We begin with initial dataflow facts ι that safely approximate ($\sqsupseteq$) all initial stores σ_1
  - By local soundness, each transfer function when given safe input information yields safe output information
  - By induction, any fixed point of the analysis is sound

Global Soundness

- Theorem (Global Soundness)
  - Assume that $\forall T \in traces(S_*)\ \alpha_{DF}(T,0) \sqsubseteq \iota$ and that analysis DF is monotone and locally sound with respect to α_{DF}
  - Then for any fixed point DF_{fix} of DF on program S_*, $\forall T \in traces(S_*)$ $\forall t \in times(T)$ we have $\alpha_{DF}(T,t) \sqsubseteq DF_{fix}(pp(T,t))$
- Proof: For each trace *T* we do induction on t
  - Induction hypothesis: $\alpha_{DF}(T,t) \sqsubseteq DF_{fix}(pp(T,t))$
  - Base case: t=0
    - By assumption $\alpha_{DF}(T,0) \sqsubseteq \iota = DF_{fix}(pp(T,0))$
  - Inductive case: time t and statement S_t
    - *Simplifying assumption: straight-line control flow*
    - By induction hypothesis we have $\alpha_{DF}(T,t\text{-}1) \sqsubseteq DF_{fix}(pp(T,t\text{-}1))$
    - By monotonicity of DF we have: $f_{DF}(S_t, \alpha_{DF}(T,t\text{-}1)) \sqsubseteq f_{DF}(S_t, DF_{fix}(pp(T,t\text{-}1)))$
    - By local soundness we have $\alpha_{DF}(T,t) \sqsubseteq f_{DF}(S_t, \alpha_{DF}(T,t\text{-}1))$
    - By transitivity we get $\alpha_{DF}(T,t) \sqsubseteq f_{DF}(S_t, DF_{fix}(pp(T,t\text{-}1)))$
    - But $f_{DF}(S_t, DF_{fix}(pp(T,t\text{-}1))) = DF_{fix}(pp(T,t))$
    - So we have $\alpha_{DF}(T,t) \sqsubseteq DF_{fix}(pp(T,t))$

Abstraction for Reaching Definitions

- $\alpha_{RD}(<p_1,m_1>\ldots<p_n,m_n>,t) =$

  $\{\ (x, p_k) \mid x \in FV(S_*)$

  and $k < t$

  and $stmt(p_k) = [x := a]$

  and $\forall j,\ k<j<t\ stmt(p_j) \neq [x := a']\}$

Local Soundness for Reaching Definitions

To show:

if $(S_i, \sigma_i) \rightarrow (S_{i+1}, \sigma_{i+1}) \in T$

and $d_{in} = \alpha_{RD}(T,i)$

and $d_{out} = f_{RD}(\mathit{first}(S_i), d_{in})$

then $\alpha_{RD}(T,i+1) \sqsubseteq d_{out}$

- Case: $S_i = [x := a]^\ell$
  - $d_{in} = \alpha_{RD}(T,i)$
  - $d_{out} = f_{RD}([x := a]^\ell, d_{in})$
    $= (\alpha_{RD}(T,i) \setminus \{(x,*)\}) \cup \{(x,\ell)\}$
  - Lemma: $\alpha_{RD}(T,i+1)$
    $= (\alpha_{RD}(T,i) \setminus \{(x,*)\}) \cup \{(x,\ell)\}$
  - So $\alpha_{RD}(T,i+1) = d_{out}$
  - Thus $\alpha_{RD}(T,i+1) \sqsubseteq d_{out}$

Abstraction for Live Variables

- $\alpha_{LV}(<p_1,m_1>\ldots<p_n,m_n>,t) =$
  $\{ x \mid x \in \mathit{FV}(\mathit{stmt}(p_k))$ where $k > t$
  and $\forall j,\ t<j<k\ \mathit{stmt}(p_j) \neq [x := a']\}$

Local Soundness
for Live Variables

To show:

if $(S_i, \sigma_i) \rightarrow (S_{i+1}, \sigma_{i+1}) \in T$
and $d_{in} = \alpha_{LV}(T, i+1)$
and $d_{out} = f_{LV}(\mathit{first}(S_{i+1}), d_{in})$
then $\alpha_{LV}(T, i) \sqsubseteq d_{out}$

Note: i and i+1 are swapped due to reverse analysis

- Case: $S_{i+1} = [x := a]^{\ell}$
  - $d_{in} = \alpha_{RD}(T, i+1)$
  - $d_{out} = f_{RD}([x := a]^{\ell}, d_{in})$
    $= (\alpha_{RD}(T, i+1) \setminus \{x\}) \cup FV(a)$
  - Lemma: $\alpha_{RD}(T, i)$
    $= (\alpha_{RD}(T, i+1) \setminus \{x\}) \cup FV(a)$
  - So $\alpha_{RD}(T, i) = d_{out}$
  - Thus $\alpha_{RD}(T, i) \sqsubseteq d_{out}$

Iterative Worklist Algorithm

Reading: NNH 2.4

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Worklist Algorithm

```
worklist = new Stack();
$\forall \ell \in \mathit{labels}(S_*)$ do
    analysis[$\ell$] = $\bot$;
$\forall \ell \in E$ do
    analysis[$\ell$] = $\iota$;
    worklist.addAll({($\ell$, $\ell_2$) | ($\ell$, $\ell_2$) $\in F$});

while (!worklist.isEmpty()) do
    ($\ell_1$, $\ell_2$) = worklist.pop();
    if $f_{\ell 1}$(Analysis[$\ell_1$]) $\not\sqsubseteq$ Analysis[$\ell_2$] then
        Analysis[$\ell_2$] = Analysis[$\ell_2$] $\sqcup$ $f_{\ell 1}$(Analysis[$\ell_1$])
        $\forall \ell_3$ where ($\ell_2$, $\ell_3$) $\in F$ do
            worklist.add(($\ell_2$, $\ell_3$));
```

Example of Worklist

```
$[a := 1]^1$
$[b := 2]^2$
while $[a < 2]^3$ do
    $[b := b * 1]^4$;
    $[a := a + 1]^5$;
$[a := b + 1]^6$;
```

Iter	**Edge**	**Worklist**	**$a_\circ$**	**$b_\circ$**
0	-	1→2	$\top$	$\top$
1	1→2	2→3	1	$\top$
2	2→3	3→4,3→6	1	2
3	3→4	4→5,3→6	1	2
4	4→5	5→3,3→6	1	2
5	5→3	3→4,3→6	$\top$	2
6	3→4	4→5,3→6	$\top$	2
7	4→5	5→3,3→6	$\top$	2
8	5→3	3→6	$\top$	2
9	3→6	-	$\top$	2

Worklist: Properties

- Correctness
  - – Implements chaotic iteration, therefore correct
- Performance
  - – Visits each node only when data changes
    - up to h = height of lattice
  - – Propagates data along control flow edges
    - up to e = max outbound edges per node
  - – Assume lattice operation cost is o
  - – Overall, O(h*e*o)

Worklist Algorithm

h: *height of lattice and max times any node can change*

n: number of nodes

e: number of edges

o: cost of data flow operations

```
worklist = new Stack();
$\forall \ell \in labels(S_*)$ do
    analysis[$\ell$] = $\bot$;
$\forall \ell \in E$ do
    analysis[$\ell$] = $\iota$;
    worklist.addAll({($\ell$, $\ell_2$) | ($\ell$, $\ell_2$) $\in F$});

while (!worklist.isEmpty()) do
    ($\ell_1$, $\ell_2$) = worklist.pop();                                may execute O(h*e) times
    if $f_{\ell 1}$(Analysis[$\ell_1$]) $\not\sqsubseteq$ Analysis[$\ell_2$] then          cost O(h*e*o)
                                                                           may execute O(h*n) times
        Analysis[$\ell_2$] = Analysis[$\ell_2$] $\sqcup$ $f_{\ell 1}$(Analysis[$\ell_1$])      cost O(h*n*o)
                                                                           may execute O(h*e) times
        $\forall \ell_3$ where ($\ell_2$, $\ell_3$) $\in F$ do                               cost O(h*e)
            worklist.add(($\ell_2$, $\ell_3$));                                   cost O(h*e)
```

Performance

	h	**o**	O(h*e*o)
Reaching Definitions			
Live Variables			
Constant Propagation (sets)			
Constant Propagation (lattice)			
Sign Analysis			

Performance

	h	**o**	O(h*e*o)
Reaching Definitions	n	n	n^2 * e
Live Variables	v	v	v^2 * e
Constant Propagation (sets)	∞	∞	∞ (May not terminate)
Constant Propagation (lattice)	2*v	2*v	v^2 * e
Sign Analysis	2*v	2*v	v^2 * e

Nonterminating Analysis

(Moral: make your lattices finite height!)

```
$[x := 0]^1$
while $[x < y]^2$ do
   $[x := x + 1]^3$;
$[x := 0]^4$;
```

Iter	**Position**	**x**	**y**
0	--	$\emptyset$	$\emptyset$
1	entry(1)	$\mathbb{Z}$	$\mathbb{Z}$
2	exit(1)	{0}	$\mathbb{Z}$
3	entry(2)	{0}	$\mathbb{Z}$
4	exit(2)	{0}	$\mathbb{Z}$
5	entry(3)	{0}	$\mathbb{Z}$
6	exit(3)	{1}	$\mathbb{Z}$
7	entry(2)	{0,1}	$\mathbb{Z}$
8	exit(2)	{0,1}	$\mathbb{Z}$
9	entry(3)	{0,1}	$\mathbb{Z}$
10	exit(3)	{1,2}	$\mathbb{Z}$
11	entry(2)	{0,1,2}	$\mathbb{Z}$
12	exit(2)	{0,1,2}	$\mathbb{Z}$
13	entry(3)	{0,1,2}	$\mathbb{Z}$
14	exit(3)	{1,2,3}	$\mathbb{Z}$
15	entry(2)	{0,1,2,4}	$\mathbb{Z}$
…			